Shift Radix Systems III

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Definition of shift radix systems (SRS)

For $d \in \mathbb{N}$ let $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Define the mapping

$$
\tau_r : \mathbb{Z}^d \rightarrow \mathbb{Z}^d
$$

$$
\mathbf{a} \mapsto (a_2, \ldots, a_d, -\lfloor ra \rfloor)
$$

where $\mathbf{a} = (a_1, \ldots, a_d)$. We call $\tau_r$ a shift radix system (SRS for short).

If

$$
\text{for all } \mathbf{a} \in \mathbb{Z}^d \text{ there exists } k > 0 \text{ with } \tau_r^k(\mathbf{a}) = 0
$$

we say that the SRS has the finiteness property.
Sets associated with SRS

\[ D_d := \left\{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d, \{ \tau_r^k(a) \} \_{k \geq 0} \text{ is ultimately periodic} \right\}, \]

\[ D_d^{(0)} := \left\{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0 \right\}. \]

We want to find a dense set of tiling parameters!
Integral self-affine tiles

Definition

Let $A$ be an expanding $d \times d$ integer matrix and $\mathcal{D} \subset \mathbb{Z}^d$. The non-empty compact set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ defined by

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + d)$$

is called integral self-affine tile, if its $d$-dimensional Lebesgue-measure is positive.

Integral self-affine tiles were studied extensively e.g. by R. Kenyon, K. Gröchenig, A. Haas, J. Lagarias, Y. Wang, and others

W.l.o.g. we always assume that $\mathbf{0} \in \mathcal{D}$.
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W.l.o.g. we always assume that $0 \in \mathcal{D}$. 
Knuth’s twin dragon

Example

\[
A = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix},
\]
\[
\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}
\]

This self-affine tile was studied by D. Knuth in connection with number systems defined in the ring \( \mathbb{Z}[i] \).
Let $\mathcal{F}(A, D)$ be an integral self-affine tile.

- If $D$ is a complete set of cosets of $\mathbb{Z}^d/A\mathbb{Z}^d$ then it is called a standard digit set for $A$.
- If $\mathbb{Z}[D, AD, A^2D, \ldots] = \mathbb{Z}^d$ then $D$ is called a primitive digit set.

It is easy to see that standard digit sets $D$ always lead to integral self-affine tiles, i.e., for a standard digit set $D$ we always have $\lambda_d(\mathcal{F}(A, D)) > 0$.

The converse is not true (see recent papers by Ka-Sing Lau, Hui Rao, and others)!
Let $\mathcal{F}(A, \mathcal{D})$ be an integral self-affine tile.

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Let $\mathcal{F}(A, D)$ be an integral self-affine tile. There exists a set $I \subset \mathbb{Z}^d$ such that $\mathcal{F} + I$ forms a tiling of $\mathbb{R}^d$.

Lagarias and Wang proved that $\mathcal{F} + \mathbb{Z}^d$ forms a multiple tiling of $\mathbb{R}^d$. In particular $\lambda_d(\mathcal{F}) \in \mathbb{N}$.

Theorem (Lagarias and Wang 1997)

Let $A$ be an integer matrix with irreducible characteristic polynomial and let $D$ be a primitive standard digit set. Then $\mathcal{F}(A, D)$ tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^n$.

Proof: Based on a Fourier analytic tiling criterion of Gröchenig and Haas and on a result of Cerveau, Conze and Raugi on transfer operators.
Let $F(A, D)$ be an integral self-affine tile. There exists a set $I \subset \mathbb{Z}^d$ such that $F + I$ forms a tiling of $\mathbb{R}^d$.

Lagarias and Wang proved that $F + \mathbb{Z}^d$ forms a multiple tiling of $\mathbb{R}^d$. In particular, $\lambda_d(F) \in \mathbb{N}$.

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**Proof:** Based on a Fourier analytic tiling criterion of Gröchenig and Haas and on a result of Cerveau, Conze and Raugi on transfer operators.
An Example

Consider $\mathcal{F}(A, D)$ with $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. 

Tame twindragon
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Definition of rational self-affine tiles

Definition

Let \( \alpha \) be a root of the characteristic polynomial and let \((r, s)\) be the signature of \( \mathbb{Q}(\alpha) \). Choose \( \mathcal{D} \subset \mathbb{Z}[\alpha] \).

- Let \( (\alpha) = \frac{a}{b} \) where \((a, b) = \mathcal{O}\) and define the representation space
  \[
  K_\alpha = \prod_{p \mid \infty \text{ or } p \mid b} K_p = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{p \mid b} K_p.
  \]

- The rational self-affine tile \( \mathcal{F} = \mathcal{F}_\alpha = \mathcal{F}(A, \mathcal{D}) \) is defined by (if its Haar measure in \( K_\alpha \) is positive).
  \[
  \alpha \mathcal{F}_\alpha = \bigcup_{d \in \mathcal{D}} (\mathcal{F}_\alpha + \Phi_\alpha(d)).
  \]

\( \Phi_\alpha \) is the diagonal embedding of \( \mathbb{Q}(\alpha) \) in \( K_\alpha \).
Definition of rational self-affine tiles

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The rational self-affine tile $F = F_\alpha = F(A, D)$ is defined by (if its Haar measure in $K_\alpha$ is positive).

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- Goal of the talk
- Integral self-affine tiles
- Rational self-affine tiles
- Intersection Tiles
- Relations to SRS
- Remarks on the proofs
Definition of rational self-affine tiles

Let $\alpha$ be a root of the characteristic polynomial and let $(r, s)$ be the signature of $\mathbb{Q}(\alpha)$. Choose $\mathcal{D} \subset \mathbb{Z}[\alpha]$.

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$\Phi_\alpha$ is the diagonal embedding of $\mathbb{Q}(\alpha)$ in $K_\alpha$. 
Remarks on the definition

Haar Measure

The Haar measure $\mu$ on $K_\alpha$ is the product of the Haar measures $\mu_p$ on $K_p$. For finite $p$ we set $\mu_p(a + p^m) = \mathcal{N}(p)^{-m}$, for infinite primes $\mu_p$ is the Lebesgue measure on $K_p$ ($K_p = \mathbb{R}$ or $\mathbb{C}$).

Embedding of $\mathbb{Q}(\alpha)$

$\mathbb{Q}(\alpha)$ is naturally embedded in $K_\alpha$ diagonally via $\Phi(\xi) = (\alpha, \ldots, \alpha)$. Moreover, $\mathbb{Q}(\alpha)$ acts multiplicatively on the ring $K_\alpha$, in particular $\xi \cdot z = \Phi_\alpha(\xi)z$.

Tiling of $\mathbb{R}^d$

One could also define $\mathcal{F}_\alpha$ as a subset of $\mathbb{R}^d$ as in the integral case. However, this would result in sets that have no nice tiling properties.
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Tiling of $\mathbb{R}^d$

One could also define $F_\alpha$ as a subset of $\mathbb{R}^d$ as in the integral case. However, this would result in sets that have no nice tiling properties.
Choose an expanding matrix \( A \in \mathbb{Q}^{d \times d} \) with irreducible characteristic polynomial.

Let \( \alpha \) be a root of the characteristic polynomial of \( A \).

Choose a basis of \( \mathbb{Q}(\alpha) \) (viewed as a vector space over \( \mathbb{Q} \)) such that the multiplication by \( \alpha \) is done by \( A \) in this vector space.

Choose a subset \( \mathcal{D} \subset \mathbb{Z}[\alpha] \) as set of digits.

An equivalent definition for a rational self-affine tile \( \mathcal{F} \subset \mathbb{K}_\alpha \) is given by the set equation

\[
A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \Phi_\alpha(d)).
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An equivalent definition for a rational self-affine tile $F \subset K_\alpha$ is given by the set equation

$$AF = \bigcup_{d \in D} (F + \Phi_\alpha(d)).$$
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Rational self-affine tiles with rational matrices

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- Let $\alpha$ be a root of the characteristic polynomial of $A$.
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Again we need special properties of the digit set.

Definition

Let $\alpha$ be expanding and $\mathcal{D} \subset \mathbb{Z}[\alpha]$ be given.

- The digit set $\mathcal{D}$ is called **primitive** if
  \[
  \langle \mathcal{D}, \alpha \mathcal{D}, \alpha^2 \mathcal{D}, \ldots \rangle_{\mathbb{Z}} = \mathbb{Z}[\alpha].
  \]

- The digit set $\mathcal{D}$ is called **standard digit set** if $\mathcal{D}$ is a complete set of residues of $\mathbb{Z}[\alpha]/\alpha \mathbb{Z}[\alpha]$.

Set $\mathbb{Z}\langle \alpha, \mathcal{D} \rangle = \langle \mathcal{D}, \alpha \mathcal{D}, \alpha^2 \mathcal{D}, \ldots \rangle_{\mathbb{Z}}$
Basic properties

Lemma

Let $\sum_{j=0}^{d} a_j x^j$ be the minimal polynomial of $\alpha$.

- $\mu_\alpha(\alpha M) = \mu_\alpha(M) \prod_{p \in S_\alpha} |\alpha|_p = a_0 \mu_\alpha(M)$.
- $\{ \Phi_\alpha(x) : x \in \mathbb{Z} \langle \alpha, D \rangle \}$ is a Delone set in $K_\alpha$.

Theorem

Let $\alpha$ and $D$ be given in a way that $D$ is a standard digit set.

- $F_\alpha$ is a compact set which is equal to the closure of its interior; its boundary has measure zero.
- $\{ F_\alpha + \Phi_\alpha(x) : x \in \mathbb{Z} \langle \alpha, D \rangle \}$ forms a multiple tiling of $K_\alpha$.
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Much harder to prove is the following tiling theorem.

**Theorem**

Let $\alpha$ be an expanding algebraic number and let $\mathcal{D}$ be a standard digit set for $\alpha$. Then $\{F + \Phi_\alpha(x) : x \in \mathbb{Z}\langle \alpha, \mathcal{D} \rangle\}$ forms a tiling of $K_\alpha$.

For primitive digit sets, we get the following immediate corollary.

**Corollary**

Let $\alpha$ be an expanding algebraic number and let $\mathcal{D}$ be a primitive, standard digit set for $\alpha$. Then $\{F + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha]\}$ forms a tiling of $K_\alpha$.

Note that for instance $\{0, 1\} \in \mathcal{D}$ implies primitivity of the digit set.
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Note that for instance $\{0, 1\} \in \mathcal{D}$ implies primitivity of the digit set.
A first example

- $\alpha = \frac{3}{2}$ and $D = \{0, 1, 2\}$.
- $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, thus $\alpha \mathcal{O} = \left(\frac{3}{2}\right)$,
- Representation space $K_{3/2} = \mathbb{R} \times \mathbb{Q}_2$.
- $F = F(3/2, \{0, 1, 2\})$ is a compact subset of $K_{3/2}$,
- $F = \text{int}(F)$ and $\mu_{3/2}(\partial F) = 0$.
- $\{F + \Phi_{3/2}(x) : x \in \mathbb{Z}[3/2]\}$, forms a tiling of $K_{3/2}$. 


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- $\alpha = \frac{3}{2}$ and $D = \{0, 1, 2\}$.
- $K = \mathbb{Q}$, $O = \mathbb{Z}$, thus $\alpha O = \frac{(3)}{2}$,
- Representation space $K_{\frac{3}{2}} = \mathbb{R} \times \mathbb{Q}_2$.
- $F = F(\frac{3}{2}, \{0, 1, 2\})$ is a compact subset of $K_{\frac{3}{2}}$,
- $F = \text{int}(F)$ and $\mu_{\frac{3}{2}}(\partial F) = 0$.
- $\{F + \Phi_{\frac{3}{2}}(x) : x \in \mathbb{Z}[\frac{3}{2}]\}$, forms a tiling of $K_{\frac{3}{2}}$. 
A first example

- $\alpha = \frac{3}{2}$ and $D = \{0, 1, 2\}$.
- $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, thus $\alpha \mathcal{O} = \frac{(3)}{(2)}$.
- Representation space $K_{\frac{3}{2}} = \mathbb{R} \times \mathbb{Q}_2$.
- $\mathcal{F} = \mathcal{F}(\frac{3}{2}, \{0, 1, 2\})$ is a compact subset of $K_{\frac{3}{2}}$.
- $\mathcal{F} = \text{int}(\mathcal{F})$ and $\mu_{\frac{3}{2}}(\partial \mathcal{F}) = 0$.
- $\{\mathcal{F} + \Phi_{\frac{3}{2}}(x) : x \in \mathbb{Z}[\frac{3}{2}]\}$, forms a tiling of $K_{\frac{3}{2}}$. 
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- $\mathcal{F} = \mathcal{F}(\frac{3}{2}, \{0, 1, 2\})$ is a compact subset of $\mathbb{K}_{\frac{3}{2}}$.
- $\overline{\mathcal{F}} = \text{int}(\mathcal{F})$ and $\mu_{\frac{3}{2}}(\partial \mathcal{F}) = 0$.
- $\{\mathcal{F} + \Phi_{\frac{3}{2}}(x) : x \in \mathbb{Z}[\frac{3}{2}]\}$, forms a tiling of $\mathbb{K}_{\frac{3}{2}}$. 
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- $\alpha = \frac{3}{2}$ and $D = \{0, 1, 2\}$.
- $K = \mathbb{Q}$, $O = \mathbb{Z}$, thus $\alpha O = \left(\frac{3}{2}\right)$.
- Representation space $K_{\frac{3}{2}} = \mathbb{R} \times \mathbb{Q}_2$.
- $F = F\left(\frac{3}{2}, \{0, 1, 2\}\right)$ is a compact subset of $K_{\frac{3}{2}}$.
- $F = \text{int}(F)$ and $\mu_{\frac{3}{2}}(\partial F) = 0$.
- $\{F + \Phi_{\frac{3}{2}}(x) : x \in \mathbb{Z}[\frac{3}{2}]\}$, forms a tiling of $K_{\frac{3}{2}}$. 
A first example - the picture

Here, an element $\sum_{j=k}^{\infty} b_j \alpha^{-j}$ of $\mathbb{Q}_2$, with $b_j \in \{0, 1\}$, is represented by $\sum_{j=k}^{\infty} b_j 2^{-j}$. 
Definition of intersection tiles

**Definition**

Let \( \mathcal{F} = \mathcal{F}(\alpha, \mathcal{D}) \) be a rational self-affine tile. For \( x \in \mathbb{Z}[\alpha] \) we call

\[
\mathcal{G}(x) = \left\{ (z_p)_{p \in S_{\alpha}} \in \mathcal{F} + \Phi_{\alpha}(x) : z_p = 0 \text{ for each } p \mid b \right\}
\]

the **intersection tile** at \( x \).

The set \( \mathcal{G}(x) \) is the intersection of \( \mathcal{F} + \Phi_{\alpha}(x) \) with

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\mathbb{K}_\infty \times \prod_{p \mid b} \{0\} \simeq \mathbb{R}^d.
\]

This justifies the terminology **intersection tile** for \( \mathcal{G}(x) \).
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the intersection tile at $x$.

The set $\mathcal{G}(x)$ is the intersection of $\mathcal{F} + \Phi_\alpha(x)$ with

$$\mathbb{K}_\infty \times \prod_{p \mid b} \{0\} \cong \mathbb{R}^d.$$ 

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Results on intersection tiles

**Lemma**

\[ \mathcal{F} \text{ is essentially made up of slices of translated copies of } \mathcal{G}(x), \quad x \in \mathbb{Z}\langle \alpha, D \rangle, \text{ i.e.,} \]

\[ \mathcal{F} = \bigcup_{x \in \mathbb{Z}\langle \alpha, D \rangle} (\mathcal{G}(x) - \Phi_\alpha(x)). \]

A uniformly locally finite collection \( \mathcal{C} \) of compact sets is called a **weak tiling** of \( \mathbb{K}_\infty \) if it is a covering of \( \mathbb{K}_\infty \) and if the interiors of the elements of \( \mathcal{C} \) are pairwise disjoint.

**Theorem**

Let \( \alpha \) be an expanding algebraic number and let \( D \) be a standard digit set for \( \alpha \). Then \( \{ \mathcal{G}(x) : x \in \mathbb{Z}\langle \alpha, D \rangle \} \) forms a weak tiling of \( \mathbb{K}_\infty \cong \mathbb{R}^n \).
Results on intersection tiles

**Lemma**

\[ \mathcal{F} \text{ is essentially made up of slices of translated copies of } G(x), \]
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$\mathcal{F}$ is essentially made up of slices of translated copies of $G(x)$, $x \in \mathbb{Z}\langle \alpha, D \rangle$, i.e.,

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A uniformly locally finite collection $\mathcal{C}$ of compact sets is called a weak tiling of $\mathbb{K}_\infty$ if it is a covering of $\mathbb{K}_\infty$ and if the interiors of the elements of $\mathcal{C}$ are pairwise disjoint.

Theorem

Let $\alpha$ be an expanding algebraic number and let $\mathcal{D}$ be a standard digit set for $\alpha$. Then $\{G(x) : x \in \mathbb{Z}\langle \alpha, D \rangle\}$ forms a weak tiling of $\mathbb{K}_\infty \simeq \mathbb{R}^n$. 
The intersection tiles are depicted on the bottom. Many of them are empty (e.g. $G(1) = \emptyset$)!
An example with a quadratic irrational

\[ \alpha = \frac{-1 + \sqrt{5}}{2} \text{ be a root of } 2X^2 + 2X + 3 \text{ and } D = \{0, 1, 2\}. \]

\[ K = \mathbb{Q}(\sqrt{-5}) \text{ and } \mathcal{O} = \mathbb{Z}[\sqrt{-5}] \text{ (not a principal ideal domain!).} \]

\[ \alpha \mathcal{O} = \frac{(3, 2+\sqrt{-5})}{(2, 1+\sqrt{-5})} \text{ is the prime ideal decomposition of } \alpha. \]

The representation space is therefore

\[ \mathbb{K}_\alpha = \mathbb{C} \times K_{(2, 1+\sqrt{-5})}. \]

Since \( D \) is a standard digit set the tile \( F \) is compact, the closure of its interior, and \( \partial F \) has Haar measure zero.

Since \( D \) is a primitive standard digit set, the collection

\[ \{ F + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha] \} \text{ forms a tiling of } \mathbb{K}_\alpha. \]

The intersection tiles \( G(x), x \in \mathbb{Z}[\alpha] \), form a weak tiling of \( \mathbb{C} \).
An example with a quadratic irrational

\[ \alpha = \frac{-1 + \sqrt{-5}}{2} \] be a root of \( 2X^2 + 2X + 3 \) and \( \mathcal{D} = \{0, 1, 2\} \).

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Since \( \mathcal{D} \) is a standard digit set the tile \( \mathcal{F} \) is compact, the closure of its interior, and \( \partial \mathcal{F} \) has Haar measure zero.

Since \( \mathcal{D} \) is a primitive standard digit set, the collection \( \{ \mathcal{F} + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha] \} \) forms a tiling of \( K_\alpha \).

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An example with a quadratic irrational

\( \alpha = \frac{-1+\sqrt{-5}}{2} \) be a root of \( 2X^2 + 2X + 3 \) and \( \mathcal{D} = \{0, 1, 2\} \).

\( K = \mathbb{Q}(\sqrt{-5}) \) and \( \mathcal{O} = \mathbb{Z}[\sqrt{-5}] \) (not a principal ideal domain!).

\( \alpha\mathcal{O} = \frac{(3, 2+\sqrt{-5})}{(2, 1+\sqrt{-5})} \) is the prime ideal decomposition of \( \alpha \).

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Since \( \mathcal{D} \) is a primitive standard digit set, the collection \( \{\mathcal{F} + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha]\} \) forms a tiling of \( K_\alpha \).

The intersection tiles \( \mathcal{G}(x), x \in \mathbb{Z}[\alpha], \) form a weak tiling of \( \mathbb{C} \).
An example with a quadratic irrational

- $\alpha = \frac{-1 + \sqrt{-5}}{2}$ be a root of $2X^2 + 2X + 3$ and $\mathcal{D} = \{0, 1, 2\}$.
- $K = \mathbb{Q}(\sqrt{-5})$ and $\mathcal{O} = \mathbb{Z}[\sqrt{-5}]$ (not a principal ideal domain!).
- $\alpha \mathcal{O} = \frac{(3, 2+\sqrt{-5})}{(2, 1+\sqrt{-5})}$ is the prime ideal decomposition of $\alpha$.
- The representation space is therefore $K_\alpha = \mathbb{C} \times K_{(2, 1+\sqrt{-5})}$.
- Since $\mathcal{D}$ is a standard digit set the tile $\mathcal{F}$ is compact, the closure of its interior, and $\partial \mathcal{F}$ has Haar measure zero.
- Since $\mathcal{D}$ is a primitive standard digit set, the collection $\{\mathcal{F} + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha]\}$ forms a tiling of $K_\alpha$.
- The intersection tiles $\mathcal{G}(x), x \in \mathbb{Z}[\alpha]$, form a weak tiling of $\mathbb{C}$. 
An example with a quadratic irrational

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- Since \( \mathcal{D} \) is a standard digit set the tile \( \mathcal{F} \) is compact, the closure of its interior, and \( \partial \mathcal{F} \) has Haar measure zero.
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Since \( \mathcal{D} \) is a standard digit set the tile \( \mathcal{F} \) is compact, the closure of its interior, and \( \partial \mathcal{F} \) has Haar measure zero.

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The intersection tiles \( \mathcal{G}(x), x \in \mathbb{Z}[\alpha] \), form a weak tiling of \( \mathbb{C} \).
An example with a quadratic irrational

- $\alpha = \frac{-1+\sqrt{-5}}{2}$ be a root of $2X^2 + 2X + 3$ and $D = \{0, 1, 2\}$.
- $K = \mathbb{Q}(-5)$ and $O = \mathbb{Z}[\sqrt{-5}]$ (not a principal ideal domain!).
- $\alpha O = \frac{(3, 2+\sqrt{-5})}{(2, 1+\sqrt{-5})}$ is the prime ideal decomposition of $\alpha$.
- The representation space is therefore $K_\alpha = \mathbb{C} \times K_{(2, 1+\sqrt{-5})}$.
- Since $D$ is a standard digit set the tile $F$ is compact, the closure of its interior, and $\partial F$ has Haar measure zero.
- Since $D$ is a primitive standard digit set, the collection $\{F + \Phi_\alpha(x) : x \in \mathbb{Z}[\alpha]\}$ forms a tiling of $K_\alpha$.
- The intersection tiles $G(x), x \in \mathbb{Z}[\alpha]$, form a weak tiling of $\mathbb{C}$.
The picture to this example

The tile $\mathcal{F}$ and a patch of the weak intersection tiling.
When is $\mathcal{G}(x)$ nonempty?

Choose $m$ in a way that $D \subset \alpha^m \mathbb{Z} [\alpha^{-1}]$ and set

$$\Lambda_{\alpha,m} = \mathbb{Z} [\alpha] \cap \alpha^{m-1} \mathbb{Z} [\alpha^{-1}] \quad (m \in \mathbb{Z}).$$

**Lemma**

*For every $x \in \mathbb{Z} [\alpha]$, we have*

$$\mathcal{G}(x) = \Phi_{\alpha}(x) + \left\{ \sum_{j=1}^{\infty} \Phi_{\alpha}(d_j \alpha^{-j}) : d_j \in D, \right. \left. \alpha^k x + \sum_{j=1}^{k} d_j \alpha^{k-j} \in \Lambda_{\alpha,m} \text{ for all } k \geq 0 \right\}.$$  

*In particular, we have $\mathcal{G}(x) = \emptyset$ for all $x \in \mathbb{Z} [\alpha] \setminus \Lambda_{\alpha,m}$.***
Particular digit sets

**Theorem**

Let $\alpha$ be expanding, let $\mathcal{D}$ be a standard digit set, and choose $m \in \mathbb{Z}$ such that $\mathcal{D} \subset \alpha^m \mathbb{Z}[\alpha^{-1}]$. If $\mathcal{D}$ contains a complete residue system of $\alpha^m \mathbb{Z}[\alpha^{-1}]/\alpha^{m-1} \mathbb{Z}[\alpha^{-1}]$, then

- $\mathcal{G}(x) \neq \emptyset$ if and only if $x \in \Lambda_{\alpha,m}$,
- there exists a constant $c > 0$ such that

$$
\delta_H(\mathcal{G}(x) - \Phi_\infty(x), \mathcal{G}(y) - \Phi_\infty(y)) \leq c \max_{p|\infty} |\alpha^{-k}|_p
$$

for all $x, y \in \Lambda_{\alpha,m}$ with $x - y \in \Lambda_{\alpha,m-k}$, $k \geq 0$. 
(Again) the same picture again

The last theorem is valid here for $m = 0$, moreover, we have $(2) | b$. 
The intersection tiles \( G(0) \) and \( G(2^k) \), \( 1 \leq k \leq 9 \), for \( \alpha = \frac{1+\sqrt{-5}}{2} \) and \( \mathcal{D} = \{0, 1, 2\} \).
If the condition does not hold

If the condition of the theorem does not hold, the set of $x$ with nonempty $G(x)$ can be complicated.

**Example**

- Let $\alpha = \frac{4}{3}$ and $\mathcal{D} = \{0, 1, 2, \frac{1}{3}\} \subset \alpha \mathbb{Z}[\alpha^{-1}]$
- $G(x) \neq \emptyset$ holds if and only if

$$x \neq 2 \cdot 3^j - 1 \pmod{3^{j+1}} \text{ for all } 0 \leq j < k$$
The tiling corresponding to this example

The tiles $\mathcal{F} + \Phi_\alpha(x) \in \mathbb{R} \times \mathbb{Q}_3$ for $\alpha = \frac{4}{3}$, $\mathcal{D} = \{0, 1, 2, \frac{1}{3}\}$, and the corresponding intersection tiles $\mathcal{G}(x) \in \mathbb{R}$. An element $\sum_{j=k}^{\infty} b_j \alpha^{-j}$ of $\mathbb{Q}_3$, with $b_j \in \{0, 1, 2\}$, is represented by $\sum_{j=k}^{\infty} b_j 3^{-j}$. 
The definition of SRS tiles

Definition

Let \( r \in D_d \) and define the set function \( \tau_r^{-1} \) by

\[
\tau_r^{-1}\{ (x_1, \ldots, x_d) \} := \{ (x_0, \ldots, x_{d-1}) \mid \tau_r(x_0, \ldots, x_{d-1}) = (x_1, \ldots, x_d) \}.
\]

and \( \tau_r^{-1}(A \cup B) = \tau_r^{-1}(A) \cup \tau_r^{-1}(B) \).

Define the set \( (R(r) \text{ is a scaling matrix}) \)

\[
\mathcal{I}_r(x) := \bigcup_{n \geq 1} R(r)^n \tau_r^{-n}\{(x)\}.
\]

Then \( \mathcal{I}_r(x) \) is called the SRS tile associated with \( r \) at \( x \).
Examples of SRS tiles
An SRS tiling?

The SRS tiles \( \mathcal{T}_r(\mathbf{x}) \) with 
\[ \|\mathbf{x}\|_\infty \leq 2 \]
corresponding to 
\[ r = \left( \frac{3}{4}, 1 \right). \]
Let $\alpha$ be an expanding algebraic number with minimal polynomial $a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X]$, and set

$$r = \left( \frac{a_n}{a_0}, \frac{a_{n-1}}{a_0}, \ldots, \frac{a_1}{a_0} \right).$$

Theorem

There is a bijection between $x \in \Lambda_{\alpha,0}$ and $x \in \mathbb{Z}^n$, and a linear transformation $\Psi$ such that

$$G(x) = \Psi(\mathcal{I}_r(x)) \times \prod_{p \mid b} \{0\}.$$
Let $\alpha$ be an expanding algebraic number with minimal polynomial $a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$, and set

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**Theorem**

There is a bijection between $x \in \Lambda_{\alpha,0}$ and $x \in \mathbb{Z}^n$, and a linear transformation $\Psi$ such that

$$\mathcal{G}(x) = \Psi(\mathcal{I}_r(x)) \times \prod_{p|b} \{0\}.$$
Conjecture

If $\beta$ is a Pisot (unit) number of degree $n + 1$, then the Rauzy fractals\n\[
\{ \mathcal{R}(x) \mid x \in \mathbb{Z}[\beta] \cap [0, 1) \} \text{ form a tiling of } \mathbb{R}^n.
\]

Proved for several classes of Pisot units.

Conjecture

If $\rho(\mathcal{R}(r)) < 1$, then $\{ \mathcal{T}_r(x) \mid x \in \mathbb{Z}^n \}$ forms a weak tiling of $\mathbb{R}^n$.

Proved for a dense set of $r$, see above.
Pisot conjecture

Conjecture

If $\beta$ is a Pisot (unit) number of degree $n + 1$, then the Rauzy fractals $\{ R(x) \mid x \in \mathbb{Z}[\beta] \cap [0, 1) \}$ form a tiling of $\mathbb{R}^n$.

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Conjecture

If $\rho(R(r)) < 1$, then $\{ T_r(x) \mid x \in \mathbb{Z}^n \}$ forms a weak tiling of $\mathbb{R}^n$.

Proved for a dense set of $r$, see above.
Contact set

The set $\mathcal{F} \cap \bigcup_{x \in \mathbb{Z} \setminus \{0\}} (\mathcal{F} + \Phi(x))$ can be described by the contact automaton (or contact graph). We first define its states:

1. $\Phi(\mathbb{Z} \langle \alpha, D \rangle) \subseteq \Phi(\mathbb{Z}[\alpha])$ forms a lattice in $K_\alpha$, i.e., it is a discrete additive subgroup of $K_\alpha$ and there exists a compact fundamental domain $D \subset K_\alpha$, i.e., $\{D + \Phi(x) : x \in \mathbb{Z}\}$ forms a tiling of $K_\alpha$.

2. Let $\mathcal{V} = \bigcup_{k \geq 0} \mathcal{V}_k$ with
   
   $\mathcal{V}_0 = \{x \in \mathbb{Z} \langle \alpha, D \rangle \setminus \{0\} : D \cap (D + \Phi(x)) \neq \emptyset\}$,
   
   $\mathcal{V}_k = \{x \in \mathbb{Z} \langle \alpha, D \rangle \setminus \{0\} : (\alpha x + D) \cap (y + D) \neq \emptyset \text{ for } y \in \mathcal{V}_{k-1}\}$.

3. $\Phi(\mathcal{V})$ is bounded and contained in a lattice, thus $\mathcal{V}$ is finite.
The set $\mathcal{F} \cap \bigcup_{x \in \mathbb{Z} \setminus \{0\}} (\mathcal{F} + \Phi(x))$ can be described by the contact automaton (or contact graph). We first define its states:

- $\Phi(\mathbb{Z} \langle \alpha, D \rangle) \subseteq \Phi(\mathbb{Z}[\alpha])$ forms a lattice in $\mathbb{K}_{\alpha}$, i.e., it is a discrete additive subgroup of $\mathbb{K}_{\alpha}$ and there exists a compact fundamental domain $D \subset \mathbb{K}_{\alpha}$, i.e., $\{D + \Phi(x) : x \in \mathbb{Z} \setminus \{0\}\}$ forms a tiling of $\mathbb{K}_{\alpha}$.

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The set $\mathcal{F} \cap \bigcup_{x \in 3 \setminus \{0\}} (\mathcal{F} + \Phi(x))$ can be described by the contact automaton (or contact graph). We first define its states:

- $\Phi(\mathbb{Z}\langle \alpha, D \rangle) \subseteq \Phi(\mathbb{Z}[[\alpha]])$ forms a lattice in $\mathbb{K}_\alpha$, i.e., it is a discrete additive subgroup of $\mathbb{K}_\alpha$ and there exists a compact fundamental domain $D \subset \mathbb{K}_\alpha$, i.e., $\{D + \Phi(x) : x \in 3\}$ forms a tiling of $\mathbb{K}_\alpha$.

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- $\Phi(\mathcal{V})$ is bounded and contained in a lattice, thus $\mathcal{V}$ is finite.
Contact automaton and contact matrix

- The contact automaton $A$ has set of states $V$ and a transition

  $$x \xrightarrow{d} y \text{ iff } \alpha x + d = y + d' \text{ for some } d' \in D \ (x, y \in V, \ d \in D).$$

  All states are initial, and $V_0$ is the set of terminal states.

- The set

  $$F_k = \bigcup_{d_1, \ldots, d_k \in D} \left( \sum_{j=1}^{k} \Phi(d_j \alpha^{-j}) + \alpha^{-k} \cdot D \right)$$

  is an approximation of $F$, and $F_k \cap (F_k + \Phi(x)) \neq \emptyset$ for $x \in \mathbb{Z}<\alpha, D> \setminus \{0\}$ iff $d_1 \cdots d_k$ is recognised by $A$.

- The contact matrix $C$ is the incidence matrix of $A$. 
Contact automaton and contact matrix

- The contact automaton $\mathcal{A}$ has set of states $\mathcal{V}$ and a transition

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- The **contact matrix** $\mathbf{C}$ is the incidence matrix of $\mathcal{A}$. 
We get the following tiling criterion in terms of the contact matrix.

**Proposition**

\[ \{ F + \Phi(x) : x \in \mathbb{Z} \} \text{ forms a tiling of } K_\alpha \text{ iff } \rho(C) < a_0. \]

This can be checked algorithmically. However, it is not suited to prove our tiling theorem.

To this matter we need to use Fourier analysis.
We get the following tiling criterion in terms of the contact matrix.

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To this matter we need to use Fourier analysis.
In his famous thesis, Tate (1950) studied the following characters:

\[ \chi_p : K_p \to \mathbb{C}, \quad z_p \mapsto \begin{cases} 
\exp(-2\pi i z_p) & \text{if } K_p = \mathbb{R}, \\
\exp(-4\pi i \text{Re}(z_p)) & \text{if } K_p = \mathbb{C}, \\
\exp(2\pi i \lambda_p(\text{Tr}_{K_p: \mathbb{Q}_p}(z_p))) & \text{if } p \mid b \text{ and } p \mid p\mathcal{O},
\end{cases} \]

\[ \chi_\alpha : K_\alpha \to \mathbb{C}, \quad (z_p)_p \mapsto \prod_p \chi_p(z_p), \]

where \( \lambda_p(x) \) denotes the fractional part of \( x \in \mathbb{Q}_p \), i.e.,

\[ \lambda_p(\sum_{j=k}^{\infty} b_j p^j) = \sum_{j=k}^{-1} b_j p^j \text{ for } (b_j)_{j \geq k} \text{ with } b_j \in \{0, \ldots, p-1\}. \]
Fourier analytic tiling criterion

Let $F$ be the Fourier transform w.r.t. $D^* = \mathbb{K}_\alpha/\Phi_\alpha(3^*)$, where

$$3^* = \{ \xi \in \mathbb{Q}(\alpha) : \chi_\alpha(\Phi_\alpha(\xi x)) = 1 \text{ for all } x \in \mathbb{Z}\langle \alpha, D \rangle \}.$$ 

Then

$$F\left( \sum_{x \in \mathcal{V}} t_x \chi_{\alpha,x} \right) = (t_x)_{x \in \mathcal{V}} \quad (t_x \in \mathbb{R}),$$

where $\chi_{\alpha,x}(z) = \chi_\alpha(x \cdot z)$. Let $\hat{\mathcal{C}} = F^{-1}\mathcal{C}F$.

**Proposition**

$\{F + \Phi(x) : x \in 3\}$ forms a tiling of $\mathbb{K}_\alpha$ iff no non-constant real-valued function

$$f = \sum_{x \in \mathcal{V}} t_x \chi_{\alpha,x}, \quad t_x \in \mathbb{R},$$

satisfies $\hat{\mathcal{C}}f = |a_0| f$. 


Fourier analytic tiling criterion

Let $F$ be the Fourier transform w.r.t. $D^* = K_\alpha / \Phi_\alpha(\mathfrak{z}^*)$, where

$$\mathfrak{z}^* = \{ \xi \in \mathbb{Q}(\alpha) : \chi_\alpha(\Phi_\alpha(\xi x)) = 1 \text{ for all } x \in \mathbb{Z} \langle \alpha, D \rangle \}.$$

Then

$$F\left( \sum_{x \in \mathcal{V}} t_x \chi_{\alpha,x} \right) = (t_x)_{x \in \mathcal{V}} \quad (t_x \in \mathbb{R}),$$

where $\chi_{\alpha,x}(z) = \chi_\alpha(x \cdot z)$. Let $\hat{\mathcal{C}} = F^{-1} \mathcal{C} F$.

**Proposition**

$$\{ F + \Phi(x) : x \in \mathfrak{z} \}$$ forms a tiling of $K_\alpha$ iff no non-constant real-valued function

$$f = \sum_{x \in \mathcal{V}} t_x \chi_{\alpha,x}, \quad t_x \in \mathbb{R},$$

satisfies $\hat{\mathcal{C}} f = |a_0| f$. 
Proof of the tiling theorem I

- If \( \{ F + \Phi(x) : x \in \mathbb{Z} \} \) does not form a tiling, then there exists \( f = \sum_{x \in \mathbb{Z}} t_x \chi_{\alpha,x}, \) with \( t_x \in \mathbb{R}, \) such that \( \hat{C}f = |a_0| f, \) \( f(0) > 0, \) \( \min_{z \in \mathbb{K}_\alpha} f(z) = 0. \)

- Using the correlation function of digits

\[
  u(z) = \left| \frac{1}{|a_0|} \sum_{d \in D} \chi_{\alpha}(d \cdot z) \right|^2 (z \in \mathbb{K}_\alpha),
\]

and choosing a complete residue system \( D^* \) of \( \mathbb{Z}^*/\alpha \mathbb{Z}^* \), we can write \( \hat{C} \) as transfer operator

\[
  \hat{C}f(z) = |a_0| \sum_{d^* \in D^*} u(\alpha^{-1}(z + \Phi(d^*))) f(\alpha^{-1}(z + \Phi(d^*))).
\]
Proof of the tiling theorem I

- If \( \{\mathcal{F} + \Phi(x) : x \in \mathbb{Z}\} \) does not form a tiling, then there exists \( f = \sum_{x \in \mathbb{N}} t_x \chi_{\alpha,x} \), with \( t_x \in \mathbb{R} \), such that \( \hat{C}f = |a_0| f \), \( f(0) > 0 \), \( \min_{z \in \mathbb{K}_\alpha} f(z) = 0 \).

- Using the correlation function of digits

\[
\hat{C}f(z) = |a_0| \sum_{d^* \in \mathcal{D}^*} u(\alpha^{-1}(z + \Phi(d^*))) f(\alpha^{-1}(z + \Phi(d^*))).
\]

and choosing a complete residue system \( \mathcal{D}^* \) of \( \mathbb{Z}^*/\alpha\mathbb{Z}^* \), we can write \( \hat{C} \) as transfer operator.
Proof of the tiling theorem II

- If \( f(z) = 0 \) and \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), then
  \[ f(\alpha^{-1}(z + \Phi(d^*))) = 0. \]
  If \( f(z) = 0 \), then
  \[ u(\alpha^{-1}(z + \Phi(d^*))) > 0 \]
  for some \( d^* \in \mathcal{D}^* \).

- Let \( Y \) be a minimal compact set such that \( f(z) = 0 \) for all \( z \in Y \) and \( \alpha^{-1}(z + \Phi(d^*)) \in Y \) if \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), \( z \in Y \).

- If \( \mathcal{D} \subset \alpha^m\mathbb{Z}[\alpha^{-1}] \), then \( \mathcal{V} \subset \alpha^m\mathbb{Z}[\alpha^{-1}] \).

- It is possible to choose \( \mathcal{D}^* \) such that \( |d^*|_p \leq |\alpha^{-m}|_p \) for all \( d^* \in \mathcal{D}^*, p | b \). Then the same holds for \( Y \).

- This implies \( \prod_{p|b} \chi_p(dz_p) = 1 \) and \( \prod_{p|b} \chi_p(xz_p) = 1 \) for all \( (z_p)_p \in Y \), i.e., we can “forget about the finite places”.
Proof of the tiling theorem II

- If \( f(z) = 0 \) and \( u(\alpha^{-1}(z + \Phi(d^*)) > 0 \), then \( f(\alpha^{-1}(z + \Phi(d^*))) = 0 \). If \( f(z) = 0 \), then \( u(\alpha^{-1}(z + \Phi(d^*)) > 0 \) for some \( d^* \in D^* \).

- Let \( Y \) be a minimal compact set such that \( f(z) = 0 \) for all \( z \in Y \) and \( \alpha^{-1}(z + \Phi(d^*)) \in Y \) if \( u(\alpha^{-1}(z + \Phi(d^*)) > 0 \), \( z \in Y \).

- If \( D \subset \alpha^m\mathbb{Z}[\alpha^{-1}] \), then \( V \subset \alpha^m\mathbb{Z}[\alpha^{-1}] \).

- It is possible to choose \( D^* \) such that \( |d^*|_p \leq |\alpha^{-m}|_p \) for all \( d^* \in D^*, p | b \). Then the same holds for \( Y \).

- This implies \( \prod_{p | b} \chi_p(d z_p) = 1 \) and \( \prod_{p | b} \chi_p(x z_p) = 1 \) for all \( (z_p)_p \in Y \), i.e., we can “forget about the finite places”.
Proof of the tiling theorem II

- If $f(z) = 0$ and $u(\alpha^{-1}(z + \Phi(d^*))) > 0$, then $f(\alpha^{-1}(z + \Phi(d^*))) = 0$. If $f(z) = 0$, then $u(\alpha^{-1}(z + \Phi(d^*))) > 0$ for some $d^* \in D^*$.

- Let $Y$ be a minimal compact set such that $f(z) = 0$ for all $z \in Y$ and $\alpha^{-1}(z + \Phi(d^*)) \in Y$ if $u(\alpha^{-1}(z + \Phi(d^*))) > 0$, $z \in Y$.

- If $D \subset \alpha^m\mathbb{Z}[-1]$ , then $V \subset \alpha^m\mathbb{Z}[\alpha^{-1}]$.

- It is possible to choose $D^*$ such that $|d^*|_p \leq |\alpha^{-m}|_p$ for all $d^* \in D^*$, $p | b$. Then the same holds for $Y$.

- This implies $\prod_{p | b} \chi_p(d z_p) = 1$ and $\prod_{p | b} \chi_p(x z_p) = 1$ for all $(z_p)_p \in Y$, i.e., we can “forget about the finite places”.
Proof of the tiling theorem II

- If \( f(z) = 0 \) and \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), then \( f(\alpha^{-1}(z + \Phi(d^*))) = 0 \). If \( f(z) = 0 \), then \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \) for some \( d^* \in D^* \).

- Let \( Y \) be a minimal compact set such that \( f(z) = 0 \) for all \( z \in Y \) and \( \alpha^{-1}(z + \Phi(d^*)) \in Y \) if \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), \( z \in Y \).

- If \( D \subset \alpha^mZ[\alpha^{-1}] \), then \( V \subset \alpha^mZ[\alpha^{-1}] \).

- It is possible to choose \( D^* \) such that \( |d^*|_p \leq |\alpha^{-m}|_p \) for all \( d^* \in D^* \), \( p | b \). Then the same holds for \( Y \).

- This implies \( \prod_{p | b} \chi_p(d z_p) = 1 \) and \( \prod_{p | b} \chi_p(x z_p) = 1 \) for all \( (z_p)_p \in Y \), i.e., we can “forget about the finite places”.


Proof of the tiling theorem II

- If \( f(z) = 0 \) and \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), then \( f(\alpha^{-1}(z + \Phi(d^*))) = 0 \). If \( f(z) = 0 \), then \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \) for some \( d^* \in D^* \).

- Let \( Y \) be a minimal compact set such that \( f(z) = 0 \) for all \( z \in Y \) and \( \alpha^{-1}(z + \Phi(d^*)) \in Y \) if \( u(\alpha^{-1}(z + \Phi(d^*))) > 0 \), \( z \in Y \).

- If \( D \subset \alpha^m \mathbb{Z}[\alpha^{-1}] \), then \( V \subset \alpha^m \mathbb{Z}[\alpha^{-1}] \).

- It is possible to choose \( D^* \) such that \( |d^*|_p \leq |\alpha^{-m}|_p \) for all \( d^* \in D^* \), \( p \mid b \). Then the same holds for \( Y \).

- This implies \( \prod_{p \mid b} \chi_p(d z_p) = 1 \) and \( \prod_{p \mid b} \chi_p(x z_p) = 1 \) for all \( (z_p)_p \in Y \), i.e., we can “forget about the finite places”.
Proof of the tiling theorem III

- By Cerveau–Conze–Raugi (1996), the projection of $Y$ to $K_\infty$ contains a translate of an $\alpha$-invariant subspace $V$ of $K_\infty$, and it is contained in a finite number of translates of $V$.

- By the denseness of $\Phi(Q(\alpha))$ in $K_\alpha$, the $\Phi(\mathfrak{z}^*)$-periodicity and continuity of $f$ and the Strong Approximation Theorem, $f(z) = 0$ for all $z \in K_\alpha$. This contradicts the definition of $f$, hence $\{\mathcal{F} + \Phi(x) : x \in \mathfrak{z}\}$ forms a tiling of $K_\alpha$. 
By Cerveau–Conze–Raugi (1996), the projection of $Y$ to $K_{\infty}$ contains a translate of an $\alpha$-invariant subspace $V$ of $K_{\infty}$, and it is contained in a finite number of translates of $V$.

By the denseness of $\Phi(Q(\alpha))$ in $K_{\alpha}$, the $\Phi(\beta^*)$-periodicity and continuity of $f$ and the Strong Approximation Theorem, $f(z) = 0$ for all $z \in K_{\alpha}$. This contradicts the definition of $f$, hence $\{F + \Phi(x) : x \in \beta\}$ forms a tiling of $K_{\alpha}$. 