Shift Radix Systems II

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Shift Radix Systems

Definition (cf. Akiyama et al., 2005)

Let \( r \in \mathbb{R}^d \) and

\[
\tau_r : \mathbb{Z}^d \rightarrow \mathbb{Z}^d, \quad x = (x_1, \ldots, x_d) \rightarrow (x_2, \ldots, x_d, -\lfloor rx \rfloor).
\]

The dynamical system \((\mathbb{Z}^d, \tau_r)\) is called a shift radix system (SRS). The SRS satisfies the finiteness property if

\[
\forall x \in \mathbb{Z}^d : \exists k \in \mathbb{N} \text{ such that } \tau_r^k(x) = 0.
\]
Notations

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- For $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$, denote by $R(\mathbf{r})$ the companion matrix with characteristic polynomial $\chi_{\mathbf{r}}(x) = x^d + r_{d-1}x^{d-1} + \cdots + r_0$.
- $\mathcal{E}_d := \{ \mathbf{r} \in \mathbb{R}^d \mid \rho(R(\mathbf{r})) < 1 \}$.

Proposition

If $\mathbf{r} \in \mathcal{E}_d$, then the SRS $(\mathbb{Z}^d, \tau_\mathbf{r})$ either satisfies the finiteness property or, for all $\mathbf{x} \in \mathbb{Z}^d$, the sequence $(\tau_\mathbf{r}^n(\mathbf{x}))_{n \in \mathbb{N}}$ is ultimately periodic.
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Characterization of the finiteness property

Quadratic SRS with finiteness property
SRS-tiles

Definition

Let $\mathbf{r} \in \mathcal{E}_d$ and $\mathbf{x} \in \mathbb{Z}^d$. The set

$$\mathcal{I}_r(\mathbf{x}) = \lim_{n \to \infty} R(\mathbf{r})^n \tau_r^{-n}(\mathbf{x})$$

(limit with respect to the Hausdorff metric) is called the SRS tile associated with $\mathbf{r}$. $\mathcal{I}_r(\mathbf{0})$ is called the central SRS tile associated with $\mathbf{r}$. 
“Taking the limit” for a central tile

Let \( r = \left( \frac{4}{5}, -\frac{49}{50} \right) \).
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Examples of SRS tiles
Some SRS tiles for $r = \left( \frac{9}{10}, -\frac{11}{20} \right)$
Proposition

For each \( r \in \mathcal{E}_d \), we have

- \( \mathcal{I}_r(x) \) is compact for all \( x \in \mathbb{Z}^d \).
- The family \( \{ \mathcal{I}_r(x) \mid x \in \mathbb{Z}^d \} \) is locally finite.
- \( \mathcal{I}_r(x) \) satisfies the set equation

\[
\mathcal{I}_r(x) = \bigcup_{y \in \mathcal{R}^{-1}_r(x)} R(r) \mathcal{I}_r(y).
\]

\[
\bigcup_{x \in \mathbb{Z}^d} \mathcal{I}_r(x) = \mathbb{R}^d.
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Such set equations play a role for \textbf{S-adic tiles} (V. Berthé, W. Steiner, T., in progress).
Basic properties of SRS tiles

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For each $r \in \mathcal{E}_d$, we have

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Such set equations play a role for $S$-adic tiles (V. Berthé, W. Steiner, T., in progress).
Compactness: Hausdorff limits are closed by definition. Boundedness follows from the contractivity of $R(r)$.

Local finiteness: $T_r(x)$ is bounded (uniformly in $x$) and the set of “base points” $\mathbb{Z}^d$ is a lattice.

Set equation: Follows immediately from the definition of the tiles. Just put one of the $R(r)$ of the product outside the limit.

Covering of $\mathbb{R}^d$: The lattice $\mathbb{Z}^d$ is obviously contained in the union. Thus, by the set equation, the same is true for $R(r)^k \mathbb{Z}^d$. Contractivity of $R(r)$, compactness, and local finiteness yield the result.
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Periodic points

**Definition**
For \( r \in \mathbb{R}^d \), a point \( z \in \mathbb{Z}^d \) is called purely periodic (with respect to \( \tau_r \)) if \( \tau_r^k(z) = z \) for some \( k \geq 1 \).

**Proposition**
For each \( r \in \mathcal{E}_d \), there exist only finitely many purely periodic points. \( 0 \) is the only purely periodic point if and only if \((\mathbb{Z}^d, \tau_r)\) has the finiteness property.

**SRS tiles and the origin**
Let \( r \in \mathcal{E}_d \).
- \( 0 \in \mathcal{T}_r(x) \) if and only if \( x \) is purely periodic.
- \((\mathbb{Z}^d, \tau_r)\) has the finiteness property if and only if \( 0 \in \mathcal{T}_r(0) \setminus \bigcup_{x \neq 0} \mathcal{T}_r(x) \) is an inner point of the central tile.
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Idea of proof

Pure periodicity of $x$ implies $0 \in \mathcal{I}_r(x)$.

- $x = \tau_r^{kp}(x)$ by assumption.
- Contractivity of $R(r)$ implies that $0 = \lim_{p \to \infty} R(r)^{kp}x \in \mathcal{I}_r(x)$.

$0 \in \mathcal{I}_r(x)$ implies pure periodicity of $x$

- By the set equation there is a sequence $(z_n)_{n \geq 1}$ with $z_n = \tau_r^{-n}(x)$ and $0 \in R(r)^n \mathcal{I}_r(z_n)$.
- Thus $0 \in \mathcal{I}_r(z_n)$. Thus by the local finiteness there are $n, k$ such that $z_n = z_{n+k}$.
- Thus $x = \tau_r^k(x)$ by the definition of $(z_n)_{n \geq 1}$. 
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Note

An SRS tile is not necessarily the closure of its interior!

Example

Set $r = \left( \frac{9}{10}, -\frac{11}{20} \right)$. The points $z_0 = (-1, -1)$, $z_1 = (-1, 1)$, $z_2 = (1, 2)$, $z_3 = (2, 1)$, $z_4 = (1, -1)$ are purely periodic:

$$
\tau_r : \ z_0 \dashrightarrow z_1 \dashrightarrow z_2 \dashrightarrow z_3 \dashrightarrow z_4 \dashrightarrow z_0.
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Furthermore, $\tau_r^{-k}(z_0) = \{z_{(k \mod 5)}\}$ and thus

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Problem: Find criteria for $\mathcal{T}_r(x) = \text{int}\left( \mathcal{T}_r(x) \right)$.
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Problem: Find criteria for \( \mathcal{I}_r(x) = \overline{\text{int}(\mathcal{I}_r(x))} \).
Tiles associated with an expanding polynomial

Definition (cf. Kátai, Kőrnyei)

Let $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be an expanding polynomial ($\Rightarrow |a_0| \geq 2$) and $B$ the transposed companion matrix with characteristic polynomial $A$.

$$F := \left\{ t \in \mathbb{R}^d \left| t = \sum_{i=0}^{\infty} B^{-i}(c_i, 0, \ldots, 0)^T, c_i \in \mathcal{N} \right. \right\}$$

($\mathcal{N} = \{0, \ldots, |a_0| - 1\}$) is called self-affine tile associated with $A$.

Theorem

- $F$ is compact and self-affine.
- $F$ is the closure of its interior.
- $\{x + F \mid x \in \mathbb{Z}^d\}$ induces a (multiple) tiling of $\mathbb{R}^d$. 
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Note that $\mathcal{F} = \bigcup_{c \in \mathbb{N}} B^{-1}(\mathcal{F} + (c, 0, \ldots, 0)^T)$.

$$r = \left(\frac{1}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0}\right), \quad V = \begin{pmatrix} 1 & a_{d-1} & \cdots & a_1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{d-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$ 

**Theorem**

*For all $x \in \mathbb{Z}^d$, we have*

$$\mathcal{F} = V \mathcal{I}_r(0),$$

$$x + \mathcal{F} = V \mathcal{I}_r(V^{-1}(x)).$$
Relation to SRS tiles

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SRS tiles associated with expanding polynomials

$X^2 + 2X + 2, \ r = (1/2, 1)$

$2X^2 + 3X + 3, \ r = (2/3, 1)$
(Non-monic) Canonical Number Systems

Definition

Let $A = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, $a_0 \geq 2$, $a_d \neq 0$, $Q = \mathbb{Z}[x]/A\mathbb{Z}[x]$, and $\mathcal{N} = \{0, \ldots, a_0 - 1\}$. If for each $P \in Q$, $P = d_0 + d_1 X + \cdots + d_\ell X^\ell$

then we call $(A, \mathcal{N})$ a canonical number system (CNS, for short).

This extends number systems with rational bases in the sense of Akiyama, Frougny and Sakarovitch (see later).

Interesting problem: Language of the representations.
(Non-monic) Canonical Number Systems

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The theorem given in the first lecture extends in a natural way to non-monic CNS.

**Theorem**

Let $A = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, $a_0 \geq 2$, $a_d \neq 0$, and $\mathcal{N} = \{0, \ldots, a_0 - 1\}$. Then the following assertions hold.

- If $A$ is expanding then $\tau_r$ with $r = \left( \frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0} \right)$ has only ultimately periodic orbits.
- The pair $(A, \mathcal{N})$ is a CNS if and only if $r = \left( \frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0} \right)$ has the finiteness property.
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- The pair $(A, \mathcal{N})$ is a CNS if and only if $r = \left( \frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0} \right)$ has the finiteness property.
**Brunotte Basis and Brunotte Module**

**Definition**

The **Brunotte basis modulo** $A$ is defined by $\{W_0, \ldots, W_{d-1}\}$ with

$$W_0 = a_d \quad \text{and} \quad W_k = \lambda W_{k-1} + a_{d-k} \quad \text{for} \quad 1 \leq k \leq d - 1.$$ 

The **Brunotte module** $\Lambda_A$ is the $\mathbb{Z}$-submodule of $\mathbb{Q}$ generated by the Brunotte basis. The representation mapping with respect to the Brunotte basis is denoted by

$$\Psi_A : \Lambda_A \to \mathbb{Z}^d, \quad P = \sum_{k=0}^{d-1} z_k W_k \mapsto (z_0, \ldots, z_{d-1})^t.$$
Tiles associated with expanding polynomials

Generalizing the construction of Kátai and Környei we can attach a family \( \{ G_A \mid A \in \mathbb{Q} \} \) of tiles.

These tiles are no longer self similar! Indeed, we have to keep track of the complicated language of representations.

**Theorem**

Let \( A = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x], \ a_0 \geq 2, \ a_d \neq 0, \) be an expanding polynomial, and \( r = (\frac{a_d}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0}) \). Then

\[
G_A(\psi_A^{-1}(z)) = V T_r(z) \quad \text{for all} \ z \in \mathbb{Z}^d.
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- The collection $\{G_A(P) \mid P \in \Lambda_A\}$ forms a weak $m$-tiling of $\mathbb{R}^d$ for some $m \geq 1$.
- If $(A, \mathcal{N})$ is a CNS, then $\{G_A(P) \mid P \in \Lambda_A\}$ forms a weak tiling of $\mathbb{R}^d$. 
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Rational bases (Akiyama, Frougny, and Sakarovitch)

- Fix \( p, q \in \mathbb{N} \) with \( p > q \geq 1 \).
- For \( N \in \mathbb{N} \) define the sequence \((N_i)_{i \geq 0}\) by
  \[
  N_0 := N, \quad qN_i = pN_{i+1} + a_i
  \]
  with \( a_i \in A := \{0, \ldots, p-1\} \).
- Each \( N \) admits a representation of the form
  \[
  N = \sum_{i=0}^{k} \frac{a_i}{q} \left( \frac{p}{q} \right)^i,
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### Some Examples of $\frac{3}{2}$-expansions

<table>
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- $L_{p/q}$ is not the full shift.
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Figure: The addition of one in the $\frac{3}{2}$ number system
The representation tree
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Let \( w(n) \) and \( W(n) \) be the lexicographically smallest and largest word starting at \( n \), respectively.

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Application of $\frac{p}{q}$-expansions: Mahler's Problem

- Original Problem: Distribution of $\left(\frac{3}{2}\right)^n \pmod{1}$.
- Generalization: $I \subset [0, 1]$,

$$Z_{\frac{p}{q}}(I) = \left\{ z \in \mathbb{R} : \left\{ z \left(\frac{p}{q}\right)^n \right\} \text{ stays eventually in } I \right\}$$

Problem: Find large $I$ with $Z_{\frac{p}{q}}(I) = \emptyset$ and small $I$ with $Z_{\frac{p}{q}}(I) \neq \emptyset$.

(Akiyama et al., 2008) If $p \geq 2q + 1$ there exists $Y_{\frac{p}{q}}$ of Lebesgue measure $\frac{p}{q}$ such that $Z_{\frac{p}{q}}(Y_{\frac{p}{q}})$ is countably infinite.
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Relation to SRS

- The dynamical system generating the $\frac{p}{q}$-expansions is (conjugate to) $\tau_{-q/p}$.

- The tiling $\{T_{-q/p}(N) \mid N \in \mathbb{Z}\}$ consists of (possibly degenerate) intervals with infinitely many different lengths.

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Tiles associated with a Pisot number

A Pisot number is an algebraic integer $\beta > 1$ with $|\beta_j| < 1$ for every conjugate $\beta_j$ of $\beta$. Write the minimal polynomial of $\beta$ as

$$(X - \beta)(X^d + r_{d-1}X^{d-1} + \cdots + r_0X^0) \in \mathbb{Z}[X].$$

Let $r = (r_0, \ldots, r_{d-1})$. For every $x \in \mathbb{Z}^d$, the SRS tile associated with $\beta$ is the set $\mathcal{T}_r(x) = \lim_{n \to \infty} R(r)^n \tau_r^{-n}(x)$,

$$R(r) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & -r_{d-2} & -r_{d-1} \end{pmatrix}, \quad \tau_r(z) = R(r)z + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \{rz\} \end{pmatrix}. $$
Relation to $\beta$-expansions and the $\beta$-transformation

The $\beta$-expansion of $z \in [0, 1)$ is given by the $\beta$-transformation

$$T_\beta : [0, 1) \to [0, 1), \ z \mapsto \{\beta z\} = \beta z - \lfloor \beta z \rfloor,$$

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with $b_n = \lfloor \beta T_\beta^{n-1}(z) \rfloor$.

**Lemma**

We have $\{r_\tau^n(z)\} = T_\beta^n(\{rz\})$ for all $n \geq 0$.

The map $f : \mathbb{Z}^d \to \mathbb{Z}[\beta] \cap [0, 1), \ z \mapsto \{rz\}$ is a bijection. Hence, the restriction of $T_\beta$ to $\mathbb{Z}[\beta] \cap [0, 1)$ is conjugate to $\tau_r$.

$(\mathbb{Z}^d, \tau_r)$ has the finiteness property if and only if $\beta$ has the property (F): every $x \in \mathbb{Z}[\beta] \cap [0, 1)$ has finite $\beta$-expansion.
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(Integral) $\beta$-tiles

Let $\beta_1, \ldots, \beta_d$ be the Galois conjugates of $\beta, \beta_1, \ldots, \beta_r \in \mathbb{R}$, $\beta_{r+1} = \beta_{r+s+1}, \ldots, \beta_{r+s} = \beta_{r+2s} \in \mathbb{C}$, $d = r + 2s$, $x^{(j)}$ be the corresponding conjugate of $x \in \mathbb{Q}(\beta), 1 \leq j \leq d$,

$$\Phi_\beta : \mathbb{Q}(\beta) \to \mathbb{R}^d, \ x \mapsto \left( x^{(1)}, \ldots, x^{(r)}, \Re(x^{(r+1)}), \Im(x^{(r+1)}), \ldots, \Re(x^{(r+s)}), \Im(x^{(r+s)}) \right).$$

Definition (cf. Thurston (1989), Akiyama (1999))

For $x \in \mathbb{Z}[\beta] \cap [0, 1)$, the $\beta$-tile is the (compact) set

$$\mathcal{R}_\beta(x) = \lim_{n \to \infty} \Phi_\beta(\beta^n T_\beta^{-n}(x)).$$

We have $t \in \mathcal{R}_\beta(x)$ if and only if there exist $c_k \in \mathbb{Z}$ with

$$t = \Phi_\beta(x) + \sum_{k=1}^{\infty} \Phi_\beta(\beta^{k-1} c_k), \quad \frac{c_n}{\beta} + \cdots + \frac{c_1}{\beta^n} + \frac{x}{\beta^n} \in [0, 1) \ \forall n \geq 1.$$
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Definition

For $x \in \mathbb{Z}[\beta] \cap [0, 1)$, the integral $\beta$-tile is the (compact) set

$$
S_\beta(x) = \lim_{n \to \infty} \Phi_\beta(\beta^n(T_\beta^{-n}(x) \cap \mathbb{Z}[\beta])).
$$

We have $t \in S_\beta(x)$ if and only if there exist $c_k \in \mathbb{Z}$ with

$$
t = \Phi_\beta(x) + \sum_{k=1}^\infty \Phi_\beta(\beta^{k-1}c_k), \quad \frac{c_n}{\beta} + \cdots + \frac{c_1}{\beta^n} + \frac{x}{\beta^n} \in [0, 1) \cap \mathbb{Z}[\beta] \ \forall n.
$$
Relation between SRS tiles and integral $\beta$-tiles

**Theorem**

Let $x^d + r_{d-1}x^{d-1} + \cdots + r_0 = (X - \beta_j)(X^{d-1} + q_{d-2}^{(j)}X^{d-2} + \cdots + q_0^{(j)}), \ 1 \leq j \leq d,$

$$U = \begin{pmatrix} q_0^{(1)} & q_1^{(j)} & \cdots & q_{d-2}^{(1)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_0^{(r)} & q_1^{(j)} & \cdots & q_{d-2}^{(r)} & 1 \\ \Re(q_0^{(r+1)}) & \Re(q_1^{(r+1)}) & \cdots & \Re(q_{d-2}^{(r+1)}) & 1 \\ \Im(q_0^{(r+1)}) & \Im(q_1^{(r+1)}) & \cdots & \Im(q_{d-2}^{(r+1)}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Re(q_0^{(r+s)}) & \Re(q_1^{(r+s)}) & \cdots & \Re(q_{d-2}^{(r+s)}) & 1 \\ \Im(q_0^{(r+s)}) & \Im(q_1^{(r+s)}) & \cdots & \Im(q_{d-2}^{(r+s)}) & 0 \end{pmatrix} \in \mathbb{R}^{d \times d},$$

$I_d$ be the identity matrix. For every $x \in \mathbb{Z}^d$, we have

$$S_\beta(\{rx\}) = U(T(r) - \beta I_d)T_r(x).$$
SRS tiles associated with Pisot numbers

\[ \beta^3 = \beta^2 + \beta + 1, \quad r = \left( \frac{1}{\beta}, \beta - 1 \right) \]

\[ \beta^3 = 2\beta^2 + 2\beta + 2, \quad r = \left( \frac{2}{\beta}, \beta - 2 \right) \]

The integral \( \beta \)-tiles are given by

\[ S_\beta(\{rx\}) = U(R(r) - \beta I_d)T(x), \]

but the “centers” of the integral \( \beta \)-tiles are given by

\[ \Phi_\beta(\{rx\}) = U(\tau_r(x) - \beta x) = U(R(r) - \beta I_d)x + U(0, \ldots, 0, \{rx\})^t. \]
Properties of $\beta$-tiles

If $\beta$ is a Pisot unit ($\beta^{-1} \in \mathbb{Z}[\beta]$), then

- $\mathcal{R}_\beta(x) = S_\beta(x)$ for every $x \in \mathbb{Z}[\beta] \cap [0, 1)$,
- we have only finitely many tiles up to translation,
- the boundary of each tile has zero Lebesgue measure,
- each tile is the closure of its interior,
- $\{S_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0, 1]\}$ forms a multiple tiling of $\mathbb{R}^d$,
- $\{S_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0, 1]\}$ forms a tiling if (F) holds,
- $\{S_\beta(x) \mid x \in \mathbb{Z}[\beta] \cap [0, 1]\}$ forms a tiling iff (W) holds: for every $x \in \mathbb{Z}[\beta] \cap [0, 1)$ and every $\varepsilon > 0$, there exists some $y \in [0, \varepsilon)$ with finite $\beta$-expansion such that $x + y$ has finite $\beta$-expansion,

Tiling properties

Definition

Let \( r = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \) be such that \( R(r) \) is contracting. The family \( \{ T_r(x) \mid x \in \mathbb{Z}^d \} \) forms a weak \( m \)-tiling of \( \mathbb{R}^d \) if every point of \( \mathbb{R}^d \) is contained in at least \( m \) different tiles \( T_r(x) \) and no point is in the interior of \( m + 1 \) different tiles \( T_r(x) \).

Theorem

*The family \( \{ T_r(x) \mid x \in \mathbb{Z}^d \} \) forms a weak \( m \)-tiling of \( \mathbb{R}^d \) for some \( m \geq 1 \) if one of the following conditions hold.*

- \( r \in \mathbb{Q}^d \),
- \( (X - \beta)(X^d + r_{d-1}X^{d-1} + \cdots + r_0) \in \mathbb{Z}[X] \) for some \( \beta > 1 \),
- \( r_0, \ldots, r_{d-1} \) are algebraically independent over \( \mathbb{Q} \).

*If \( (\mathbb{Z}^d, \tau_r) \) satisfies the finiteness property, then \( m = 1 \).*
(m-)exclusive points

**Definition**

\( t \in \mathbb{R}^d \) is an **exclusive point** if
\[
\# \{ x \in \mathbb{Z}^d \mid t \in \mathcal{T}_r(x) \} = 1.
\]

**Lemma**

Let \( R = \sum_{k=1}^{\infty} \| R(r)^k (0, \ldots, 0, 1)^t \| \). If, for some \( z \in \mathbb{Z}^d, n \geq 0, \)
\[
\# \{ \tau_r^n (z + y) \mid y \in \mathbb{Z}^d, \| y \| \leq R \} = 1,
\]
then \( R(r)^n z \) is an exclusive point.

**Lemma**

If \( r \) satisfies one of the conditions of the theorem and there exists an exclusive point, then \( \{ \mathcal{T}_r(x) \mid x \in \mathbb{Z}^d \} \) forms a weak \((1-)\)tiling.
**(m-)exclusive points**

**Definition**

\( t \in \mathbb{R}^d \) is an **\( m \)-exclusive point** if \( \#\{ x \in \mathbb{Z}^d \mid t \in T_r(x) \} = m \).

**Lemma**

Let \( R = \sum_{k=1}^{\infty} \| R(r)^k (0, \ldots, 0, 1)^t \| \). If, for some \( z \in \mathbb{Z}^d, n \geq 0 \),

\[
\#\{ \tau_r^n(z + y) \mid y \in \mathbb{Z}^d, \| y \| \leq R \} = m,
\]

then \( R(r)^n z \) is an **\( m' \)-exclusive point** with \( 1 \leq m' \leq m \).

**Lemma**

If \( r \) satisfies one of the conditions of the theorem and there exists an \( m \)-exclusive point, then \( \{ T_r(x) \mid x \in \mathbb{Z}^d \} \) forms a **weak \( m' \)-tiling** with \( 1 \leq m' \leq m \).
**Definition**

For \( r = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d \), the \( \alpha \)-SRS \( (\mathbb{Z}^d, \tau_{r,\alpha}) \) is defined by

\[
\tau_{r,\alpha} : \mathbb{Z}^d \to \mathbb{Z}^d, \quad x = (x_0, \ldots, x_{d-1}) \mapsto (x_1, \ldots, x_{d-1}, -\lfloor rx + \alpha \rfloor).
\]

For every \( x \in \mathbb{Z}^d \), the \( \alpha \)-SRS tile is defined by

\[
T_{r,\alpha}(x) = \lim_{n \to \infty} R(r)^n \tau_{r,\alpha}^{-n}(x).
\]

An \( 1/2 \)-SRS is also called symmetric SRS.

**Theorem (Kalle and Steiner (2011))**

Let \( \beta \) be the smallest Pisot number \( (\beta^3 = \beta + 1) \), \( r = (1/\beta, \beta) \), or the Tribonacci number \( (\beta^3 = \beta^2 + \beta + 1) \), \( r = (1/\beta, \beta - 1) \), then \( \{ T_{r,1/2}(x) \mid x \in \mathbb{Z}^2 \} \) forms a 2-tiling of \( \mathbb{R}^2 \).
Double tiling for a symmetric SRS

\[ \beta^3 = \beta^2 + \beta + 1, \quad r = \left(\frac{1}{\beta}, \beta - 1\right), \quad \alpha = 1/2 \]
Tiling for a symmetric SRS

$$\beta^3 = 2\beta^2 - \beta + 1, \ r = (1/\beta, \beta - 2), \ \alpha = 1/2$$