Shift Radix Systems I

J. M. Thuswaldner

Department of Mathematics and Statistics
University of Leoben
Austria

Liège, June 2011
Definition (S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, J. T.)

For $d \in \mathbb{N}$ let $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Call the mapping

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$$

$$\mathbf{a} \mapsto (a_2, \ldots, a_d, -\lfloor \mathbf{r} \cdot \mathbf{a} \rfloor)$$

a shift radix system (SRS); here $\mathbf{a} = (a_1, \ldots, a_d)$.

If

for all $\mathbf{a} \in \mathbb{Z}^d$ there exists $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.

we say that the SRS $\tau_{\mathbf{r}}$ has the finiteness property.
The definition

Definition (S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, J. T.)

For \( d \in \mathbb{N} \) let \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d \). Call the mapping

\[
\tau_r : \mathbb{Z}^d \to \mathbb{Z}^d
\]

\[
\mathbf{a} \mapsto (a_2, \ldots, a_d, -\lfloor ra \rfloor)
\]

a shift radix system (SRS); here \( \mathbf{a} = (a_1, \ldots, a_d) \).

If for all \( \mathbf{a} \in \mathbb{Z}^d \) there exists \( k > 0 \) with \( \tau_r^k(\mathbf{a}) = 0 \),

we say that the SRS \( \tau_r \) has the finiteness property.
How does that work?

Let \( r = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
\begin{align*}
(5, -3) &\overset{\tau_r}{\rightarrow} (-3, -4) \overset{\tau_r}{\rightarrow} (-4, 1) \overset{\tau_r}{\rightarrow} (1, 3) \overset{\tau_r}{\rightarrow} (3, 1) \overset{\tau_r}{\rightarrow} \\
(1, -1) &\overset{\tau_r}{\rightarrow} (-1, -1) \overset{\tau_r}{\rightarrow} (-1, 1) \overset{\tau_r}{\rightarrow} (1, 1) \overset{\tau_r}{\rightarrow} (1, 0) \overset{\tau_r}{\rightarrow} \\
(0, 0) &\overset{\tau_r}{\rightarrow} (0, 0)
\end{align*}
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( r \in \mathbb{R}^d \)).
How does that work?

Let \( \mathbf{r} = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
\begin{align*}
(5, -3) &\rightarrow \tau_r (-3, -4) \rightarrow (-4, 1) \rightarrow (1, 3) \rightarrow (3, 1) \rightarrow \\
(1, -1) &\rightarrow (-1, -1) \rightarrow (-1, 1) \rightarrow (1, 1) \rightarrow (1, 0) \rightarrow \\
(0, 0) &\rightarrow (0, 0)
\end{align*}
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( \mathbf{r} \in \mathbb{R}^d \)).
Let \( \mathbf{r} = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
\begin{align*}
(5, -3) \xrightarrow{\mathbf{r}} (-3, -4) & \xrightarrow{\mathbf{r}} (-4, 1) \xrightarrow{\mathbf{r}} (1, 3) \xrightarrow{\mathbf{r}} (3, 1) \xrightarrow{\mathbf{r}} (1, -1) \xrightarrow{\mathbf{r}} (-1, -1) \xrightarrow{\mathbf{r}} (-1, 1) \xrightarrow{\mathbf{r}} (1, 1) \xrightarrow{\mathbf{r}} (1, 0) \xrightarrow{\mathbf{r}} (0, 0) \xrightarrow{\mathbf{r}} (0, 0)
\end{align*}
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( \mathbf{r} \in \mathbb{R}^d \)).
How does that work?

Let \( r = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
(5, -3) \xrightarrow{\tau_r} (-3, -4) \xrightarrow{\tau_r} (-4, 1) \xrightarrow{\tau_r} (1, 3) \xrightarrow{\tau_r} (3, 1) \xrightarrow{\tau_r} \\
(1, -1) \xrightarrow{\tau_r} (-1, -1) \xrightarrow{\tau_r} (-1, 1) \xrightarrow{\tau_r} (1, 1) \xrightarrow{\tau_r} (1, 0) \xrightarrow{\tau_r} \\
(0, 0) \xrightarrow{\tau_r} (0, 0)
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( r \in \mathbb{R}^d \)).
How does that work?

Let \( r = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
\begin{align*}
(5, -3) & \xrightarrow{\tau r} (-3, -4) \xrightarrow{\tau r} (-4, 1) \xrightarrow{\tau r} (1, 3) \xrightarrow{\tau r} (3, 1) \xrightarrow{\tau r} (1, 0) \xrightarrow{\tau r} (0, 0) \\
(1, -1) & \xrightarrow{\tau r} (-1, -1) \xrightarrow{\tau r} (-1, 1) \xrightarrow{\tau r} (1, 1) \xrightarrow{\tau r} (1, 0) \xrightarrow{\tau r} (0, 0) \\
(0, 0) & \xrightarrow{\tau r} (0, 0)
\end{align*}
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( r \in \mathbb{R}^d \)).
How does that work?

Let $\mathbf{r} = \left(\frac{3}{5}, \frac{2}{5}\right)$ and start with $(5, -3)$.

$$(5, -3) \xrightarrow{\tau_r} (-3, -4) \xrightarrow{\tau_r} (-4, 1) \xrightarrow{\tau_r} (1, 3) \xrightarrow{\tau_r} (3, 1) \xrightarrow{\tau_r} (1, -1) \xrightarrow{\tau_r} (-1, -1) \xrightarrow{\tau_r} (-1, 1) \xrightarrow{\tau_r} (1, 1) \xrightarrow{\tau_r} (1, 0) \xrightarrow{\tau_r} (0, 0)$$

$(0, 0)$ is a self-loop (this is true for each parameter $\mathbf{r} \in \mathbb{R}^d$).
How does that work?

Let \( r = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
(5, -3) \xrightarrow{\tau_r} (-3, -4) \xrightarrow{\tau_r} (-4, 1) \xrightarrow{\tau_r} (1, 3) \xrightarrow{\tau_r} (3, 1) \xrightarrow{\tau_r} (3, 1)
\]

\[
(1, -1) \xrightarrow{\tau_r} (-1, -1) \xrightarrow{\tau_r} (-1, 1) \xrightarrow{\tau_r} (1, 1) \xrightarrow{\tau_r} (1, 0) \xrightarrow{\tau_r} (1, 0)
\]

\[
(0, 0) \xrightarrow{\tau_r} (0, 0)
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( r \in \mathbb{R}^d \)).
How does that work?

Let \( \mathbf{r} = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
\begin{align*}
(5, -3) & \xrightarrow{\mathbf{r}} (-3, -4) \xrightarrow{\mathbf{r}} (-4, 1) \xrightarrow{\mathbf{r}} (1, 3) \xrightarrow{\mathbf{r}} (3, 1) \xrightarrow{\mathbf{r}} \\
(1, -1) & \xrightarrow{\mathbf{r}} (-1, -1) \xrightarrow{\mathbf{r}} (-1, 1) \xrightarrow{\mathbf{r}} (1, 1) \xrightarrow{\mathbf{r}} (1, 0) \xrightarrow{\mathbf{r}} \\
(0, 0) & \xrightarrow{\mathbf{r}} (0, 0)
\end{align*}
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( \mathbf{r} \in \mathbb{R}^d \)).
How does that work?

Let $\mathbf{r} = (\frac{3}{5}, \frac{2}{5})$ and start with $(5, -3)$.

$$(5, -3) \xrightarrow{\tau r} (-3, -4) \xrightarrow{\tau r} (-4, 1) \xrightarrow{\tau r} (1, 3) \xrightarrow{\tau r} (3, 1) \xrightarrow{\tau r}$$

$$(1, -1) \xrightarrow{\tau r} (-1, -1) \xrightarrow{\tau r} (-1, 1) \xrightarrow{\tau r} (1, 1) \xrightarrow{\tau r} (1, 0) \xrightarrow{\tau r}$$

$$(0, 0) \xrightarrow{\tau r} (0, 0)$$

$(0, 0)$ is a self-loop (this is true for each parameter $\mathbf{r} \in \mathbb{R}^d$).
How does that work?

Let $\mathbf{r} = (\frac{3}{5}, \frac{-2}{5})$ and start with $(5, -3)$.

$$
(5, -3) \xrightarrow{\mathbf{r}} (-3, -4) \xrightarrow{\mathbf{r}} (-4, 1) \xrightarrow{\mathbf{r}} (1, 3) \xrightarrow{\mathbf{r}} (3, 1) \xrightarrow{\mathbf{r}} (1, 1) \xrightarrow{\mathbf{r}} (1, 0) \xrightarrow{\mathbf{r}} (0, 0)
$$

$(0, 0)$ is a self-loop (this is true for each parameter $\mathbf{r} \in \mathbb{R}^d$).
How does that work?

Let \( r = \left( \frac{3}{5}, \frac{-2}{5} \right) \) and start with \((5, -3)\).

\[
(5, -3) \xrightarrow{\tau r} (-3, -4) \xrightarrow{\tau r} (-4, 1) \xrightarrow{\tau r} (1, 3) \xrightarrow{\tau r} (3, 1) \xrightarrow{\tau r} (1, 0) \xrightarrow{\tau r} (0, 0)
\]

\((0, 0)\) is a self-loop (this is true for each parameter \( r \in \mathbb{R}^d \)).
Let $r = \left(\frac{3}{5}, \frac{-2}{5}\right)$ and start with $(5, -3)$.

$$(5, -3) \xrightarrow{\tau_r} (-3, -4) \xrightarrow{\tau_r} (-4, 1) \xrightarrow{\tau_r} (1, 3) \xrightarrow{\tau_r} (3, 1) \xrightarrow{\tau_r}$$

$$(1, -1) \xrightarrow{\tau_r} (-1, -1) \xrightarrow{\tau_r} (-1, 1) \xrightarrow{\tau_r} (1, 1) \xrightarrow{\tau_r} (1, 0) \xrightarrow{\tau_r}$$

$$(0, 0) \xrightarrow{\tau_r} (0, 0)$$

$(0, 0)$ is a self-loop (this is true for each parameter $r \in \mathbb{R}^d$).
SRS are almost linear

For $\mathbf{r} = (r_1, \ldots, r_d)$ let

$$ R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}. $$

Then

$$ \tau_\mathbf{r}(\mathbf{a}) = R(\mathbf{r}) + (0, \ldots, 0, \{\mathbf{r}\mathbf{a}\}). $$
Examples of orbits

Let \( r = (0.998, -0.997) \), hence, \( R(r) = \begin{pmatrix} 0 & 1 \\ -0.998 & 0.997 \end{pmatrix} \).

On the long run, the floor makes a big difference.
Examples of orbits

Let $r = (0.998, -0.997)$, hence, $R(r) = \begin{pmatrix} 0 & 1 \\ -0.998 & 0.997 \end{pmatrix}$.

On the long run, the floor makes a big difference.
Motivation

- SRS form generalizations of several notions of numeration.
  
  - SRS admit an interesting geometric theory that includes many well-known fractal tiles.
  
  - SRS give rise to new kinds of tilings of the $d$-dimensional real vector space.
  
  - SRS are related to many known problems as
    - periodicity of expansions w.r.t. Salem numbers (K. Schmidt, 1980),
    - the Pisot conjecture,
    - Mahler's $\frac{3}{2}$ problem and the Josephus problem.
Motivation

- SRS form generalizations of several notions of numeration.
- SRS admit an interesting geometric theory that includes many well-known fractal tiles.
- SRS give rise to new kinds of tilings of the \(d\)-dimensional real vector space.
- SRS are related to many known problems as
  - periodicity of expansions w.r.t. Salem numbers (K. Schmidt, 1980),
  - the Pisot conjecture,
  - Mahler's \(\frac{3}{2}\) problem and the Josephus problem.
Motivation

- SRS form generalizations of several notions of numeration.
- SRS admit an interesting geometric theory that includes many well-known fractal tiles.
- SRS give rise to new kinds of tilings of the $d$-dimensional real vector space.
- SRS are related to many known problems as
  - periodicity of expansions w.r.t. Salem numbers (K. Schmidt, 1980),
  - the Pisot conjecture,
  - Mahler's $\frac{3}{2}$ problem and the Josephus problem.
SRS form generalizations of several notions of numeration.

SRS admit an interesting geometric theory that includes many well-known fractal tiles.

SRS give rise to new kinds of tilings of the $d$-dimensional real vector space.

SRS are related to many known problems as
- periodicity of expansions w.r.t. Salem numbers (K. Schmidt, 1980),
- the Pisot conjecture,
- Mahler’s $\frac{3}{2}$ problem and the Josephus problem.
The $q$-ary system

Let $q \geq 2$ be given.

**Definition**

Each $n \in \mathbb{N}$ can be written as

$$n = \sum_{i=0}^{\ell} d_i q^i \quad (d_i \in \{0, \ldots, q - 1\}).$$

This is the $q$-ary representation of $n$. 
Let $q \geq 2$ be given.

**Definition**

Each $n \in \mathbb{N}$ can be written as

$$n = \sum_{i=0}^{\ell} d_i q^i \quad (d_i \in \{0, \ldots, q - 1\}).$$

This is the *q-ary representation* of $n$. 
Basic properties of \( q \)-ary representations

- Each \( q \geq 2 \) can serve as base.
- Each \( n \in \mathbb{N} \) has a unique representation if we forbid leading zeros.
- All digit strings \( \{0, \ldots, q - 1\}^* \) can occur.
- Also the reals can be represented. The set of all numbers having representations with “integer part 0” is given by

\[
\mathcal{T} = \left\{ \sum_{i \geq 1} d_i q^{-i} : 0 \leq d_i < q \right\} = [0, 1].
\]

- \( \mathcal{T} + \mathbb{Z} \) form a tiling of \( \mathbb{R} \).
Basic properties of $q$-ary representations

- Each $q \geq 2$ can serve as base.
- Each $n \in \mathbb{N}$ has a unique representation if we forbid leading zeros.
- All digit strings $\{0, \ldots, q - 1\}^*$ can occur.
- Also the reals can be represented. The set of all numbers having representations with “integer part 0” is given by

$$\mathcal{T} = \left\{ \sum_{i \geq 1} d_i q^{-i} : 0 \leq d_i < q \right\} = [0, 1].$$

- $\mathcal{T} + \mathbb{Z}$ form a tiling of $\mathbb{R}$. 
Basic properties of $q$-ary representations

- Each $q \geq 2$ can serve as base.
- Each $n \in \mathbb{N}$ has a unique representation if we forbid leading zeros.
- All digit strings $\{0, \ldots, q-1\}^*$ can occur.
- Also the reals can be represented. The set of all numbers having representations with “integer part 0” is given by

$$\mathcal{T} = \left\{ \sum_{i \geq 1} d_i q^{-i} : 0 \leq d_i < q \right\} = [0, 1].$$

- $\mathcal{T} + \mathbb{Z}$ form a tiling of $\mathbb{R}.$
Basic properties of $q$-ary representations

- Each $q \geq 2$ can serve as base.
- Each $n \in \mathbb{N}$ has a unique representation if we forbid leading zeros.
- All digit strings $\{0, \ldots, q - 1\}^*$ can occur.
- Also the reals can be represented. The set of all numbers having representations with “integer part 0” is given by

$$\mathcal{T} = \left\{ \sum_{i \geq 1} d_i q^{-i} : 0 \leq d_i < q \right\} = [0, 1].$$

- $\mathcal{T} + \mathbb{Z}$ form a tiling of $\mathbb{R}$. 
Basic properties of $q$-ary representations

- Each $q \geq 2$ can serve as base.
- Each $n \in \mathbb{N}$ has a unique representation if we forbid leading zeros.
- All digit strings $\{0, \ldots, q - 1\}^*$ can occur.
- Also the reals can be represented. The set of all numbers having representations with “integer part 0” is given by

$$\mathcal{T} = \left\{ \sum_{i \geq 1} d_i q^{-i} : 0 \leq d_i < q \right\} = [0, 1].$$

- $\mathcal{T} + \mathbb{Z}$ form a tiling of $\mathbb{R}$. 
A $q$-ary dynamical system

- $q \geq 2$, $\mathcal{D} = \{0, \ldots, q - 1\}$
- $T_q : \mathbb{Z} \to \mathbb{Z}; \ a \mapsto \frac{a-d}{q} \quad (d \in \mathcal{D})$
  
  Here $d \in \mathcal{D}$ the unique digit satisfying $(a - d)/q \in \mathbb{Z}$

- Note that $T_q^k(a) > T_q^{k-1}(a)$ for $a > 0$.

\[
a = qT_q(a) + d_0 \hspace{1cm} = q^2 T_q^2(a) + d_1 q + d_0 = \cdots \\
= q^{\ell+1} T_q^{\ell+1}(a) + d_\ell q^\ell + \cdots + d_0 \\
= d_\ell q^\ell + \cdots + d_0.
\]

$T_q$ generates the $q$-ary expansion of $a$. 
A $q$-ary dynamical system

- $q \geq 2$, $\mathcal{D} = \{0, \ldots, q - 1\}$
- $T_q : \mathbb{Z} \to \mathbb{Z}$; \( a \mapsto \frac{a - d}{q} \) \((d \in \mathcal{D})\)
  Here $d \in \mathcal{D}$ the unique digit satisfying $(a - d)/q \in \mathbb{Z}$
- Note that $T_q^k(a) > T_q^{k-1}(a)$ for $a > 0$.

$$a = qT_q(a) + d_0$$
$$= q^2T_q^2(a) + d_1 q + d_0 = \cdots$$
$$= q^{\ell+1}T_q^{\ell+1}(a) + d_\ell q^\ell + \cdots + d_0$$
$$= d_\ell q^\ell + \cdots + d_0.$$  

$T_q$ generates the $q$-ary expansion of $a$. 
Connection to SRS

Let $r = -\frac{1}{q}$. Then,

$$-\tau_{-1/q}(-a) = \left\lfloor \frac{a}{q} \right\rfloor = \frac{a - q\{\frac{a}{q}\}}{q} \in \mathbb{Z}. $$

This implies that $T_q(a) = -\tau_{-1/q}(-a)$ or

Thus $T_q$ is conjugate to the SRS mapping $\tau_{-1/q}$.
Let $r = -\frac{1}{q}$. Then,

$$-\tau_{-1/q}(-a) = \left\lfloor \frac{a}{q} \right\rfloor = \frac{a - q \{\frac{a}{q}\}}{q} \in \mathbb{Z}.$$ 

This implies that $T_q(a) = -\tau_{-1/q}(-a)$ or

Thus $T_q$ is conjugate to the SRS mapping $\tau_{-1/q}$. 
How about negative integers?

- $T_q(-1) = -1$ for all $q \geq 2$.
- $-1$ has no finite $q$-ary expansion, indeed
  $$-1 = (q - 1) + (q - 1)q + (q - 1)q^2 + \cdots$$

- The same problem occurs for each negative integer.
  - Thus by conjugacy $\tau_{-1/q}(1) = 1$.
  - $\tau_{-1/q}$ does not have the finiteness property.
  - In general we only get ultimately periodic expansions.
How about negative integers?

- \( T_q(-1) = -1 \) for all \( q \geq 2 \).
- \(-1\) has **no** finite \( q \)-ary expansion, indeed
  \[
  -1 = (q - 1) + (q - 1)q + (q - 1)q^2 + \cdots
  \]
- The same problem occurs for each negative integer.
- Thus by conjugacy \( \tau_{-1/q}(1) = 1 \).
- \( \tau_{-1/q} \) does **not** have the finiteness property.
- In general we only get **ultimately periodic** expansions.
“Better” number systems

Take \(-q\) with \(q \geq 2\) as a basis and \(\mathcal{D} = \{0, \ldots, q - 1\}\) as digits.

\[
n = d_0 + d_1(-q) + d_2(-q)^2 + \cdots + d_\ell(-q)^\ell
\]

Figure: The addition of \(\pm 1\) in the \(-3\)-ary number system

\(T_{-q}(a) = \frac{a-d}{-q}\) is conjugate to \(\tau_{1/q} \Rightarrow\) finiteness property.
We want to represent elements of $\mathbb{Z}[i]$.

**Theorem (Knuth, 1968)**

*Let $b = -1 + i$. Then each $z \in \mathbb{Z}[i]$ can be written uniquely as*

$$z = \sum_{j=0}^{\ell} d_j b^j \quad (d_j \in 0, 1).$$

$$T = \left\{ \sum_{j \geq 1} d_j b^{-j} : d_j \in \{0, 1\} \right\}$$
We want to represent elements of $\mathbb{Z}[i]$.

**Theorem (Knuth, 1968)**

Let $b = -1 + i$. Then each $z \in \mathbb{Z}[i]$ can be written uniquely as

$$z = \sum_{j=0}^{\ell} d_j b^j \quad (d_j \in 0, 1).$$

$$T = \left\{ \sum_{j \geq 1} d_j b^{-j} : d_j \in \{0, 1\} \right\}$$
Canonical number systems — Knuth’s example

We want to represent elements of $\mathbb{Z}[i]$.

**Theorem (Knuth, 1968)**

Let $b = -1 + i$. Then each $z \in \mathbb{Z}[i]$ can be written uniquely as

$$z = \sum_{j=0}^{\ell} d_j b^j \quad (d_j \in 0, 1).$$

$$T = \left\{ \sum_{j \geq 1} d_j b^{-j} : d_j \in \{0, 1\} \right\}$$
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $D = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

\[
egin{align*}
-b - 5 &+ 3(b^2 + 2b + 2) = 3b^2 + 5b + 1 & \quad \text{since } b = -1 + i \\
3b + 5 &+ (-2)(b^2 + 2b + 2) = -2b^2 - b + 1 & \quad \text{since } -b = b^2 + 1 \\
-2b - 1 &+ 1(b^2 + 2b + 2) = b^2 + 1 + 1 & \quad \text{since } 0 = 0 \\
b &+ 0(b^2 + 2b + 2) = b + \quad & \quad \text{since } 1 = 1 \\
1 &+ 0(b^2 + 2b + 2) = \quad & \quad \text{since } 1 = 1 
\end{align*}
\]

Thus $(-4 - i)_b = 10111$. 
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $\mathcal{D} = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

$$
\begin{align*}
-b - 5 &+ 3(b^2 + 2b + 2) = 3b^2 + 5b+ 1 \quad \mid \quad \frac{-b}{b} \\
3b + 5 &+ (-2)(b^2 + 2b + 2) = -2b^2 - b+ 1 \quad \mid \quad \frac{-b}{b} \\
-2b - 1 &+ 1(b^2 + 2b + 2) = b^2 + 1+ 1 \quad \mid \quad \frac{-b}{b} \\
b &+ 0(b^2 + 2b + 2) = b+ 0 \quad \mid \quad \frac{-b}{b} \\
1 &+ 0(b^2 + 2b + 2) = 1
\end{align*}
$$

Thus $(−4 − i)_b = 10111$. 
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $\mathcal{D} = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

$$
\begin{align*}
-b - 5 + & 3(b^2 + 2b + 2) = 3b^2 + 5b + 1 & \frac{-1}{b}
\\
3b + 5 + & (-2)(b^2 + 2b + 2) = -2b^2 - b + 1 & \frac{-1}{b}
\\
-2b - 1 + & 1(b^2 + 2b + 2) = b^2 + 1 + 1 & \frac{-1}{b}
\\
& b + 0(b^2 + 2b + 2) = b + 0 & \frac{-0}{b}
\\
& 1 + 0(b^2 + 2b + 2) = 1
\end{align*}
$$

Thus $(-4 - i)_b = 10111$. 
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $\mathcal{D} = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

\[
\begin{align*}
- b - 5 & + 3(b^2 + 2b + 2) = 3b^2 + 5b + 1 & \left| \frac{-1}{b} \right| \\
3b + 5 & + (-2)(b^2 + 2b + 2) = -2b^2 - b + 1 & \left| \frac{-1}{b} \right| \\
-2b - 1 & + 1(b^2 + 2b + 2) = b^2 + 1 + 1 & \left| \frac{-1}{b} \right| \\
b & + 0(b^2 + 2b + 2) = b + 0 & \left| \frac{0}{b} \right| \\
1 & + 0(b^2 + 2b + 2) = 1 & \left| \frac{0}{b} \right|
\end{align*}
\]

Thus $(−4 − i)_b = 10111$. 
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $\mathcal{D} = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

\[
\begin{align*}
- b &- 5 + 3(b^2 + 2b + 2) = 3b^2 + 5b+ 1 & \equiv & & \frac{-1}{b} \\
3b &+ 5 + (-2)(b^2 + 2b + 2) = -2b^2 - b+ 1 & \equiv & & \frac{-1}{b} \\
-2b &- 1 + 1(b^2 + 2b + 2) = b^2 + 1+ 1 & \equiv & & \frac{-1}{b} \\
b &+ 0(b^2 + 2b + 2) = b+ 0 & \equiv & & \frac{-0}{b} \\
1 &+ 0(b^2 + 2b + 2) = & & & 1
\end{align*}
\]

Thus $(−4−i)_b = 10111$. 
How to represent numbers w.r.t. $b = -1 + i$?

- $b = -1 + i$ and $D = \{0, 1\}$
- $b$ has minimal polynomial $x^2 + 2x + 2$
- We want to represent $-4 - i = -b - 5$ in terms of this number system.

\[
\begin{align*}
-b - 5 & \quad 3(b^2 + 2b + 2) = 3b^2 + 5b + 1 \quad \mid \quad \frac{-1}{b} \\
3b + 5 & \quad (-2)(b^2 + 2b + 2) = -2b^2 - b + 1 \quad \mid \quad \frac{-1}{b} \\
-2b - 1 & \quad 1(b^2 + 2b + 2) = b^2 + 1 + 1 \quad \mid \quad \frac{-1}{b} \\
b & \quad 0(b^2 + 2b + 2) = b + 0 \quad \mid \quad \frac{-0}{b} \\
1 & \quad 0(b^2 + 2b + 2) = 1
\end{align*}
\]

Thus $(-4 - i)_b = 10111$. 
Possible generalizations of canonical number systems (CNS).

- Canonical number systems in integer rings \( \mathbb{Z}_K \).
- Canonical number systems in orders \( \mathbb{Z}[\alpha] \).
- If \( P(X) \in \mathbb{Z}[X] \) is the minimal polynomial of \( \alpha \) then

\[
\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/P(X)\mathbb{Z}[X].
\]

**Definition (Pethö, 1991)**

We call \((P(X), \{0, \ldots, |P(0)| - 1\})\) a CNS if each element \( z \in \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) admits a unique expansion of the form

\[
z = d_0 + d_1X + \cdots + d_\ell X^\ell \quad (0 \leq d_j < |P(0)|).
\]
Pethö’s general definition of CNS

Possible generalizations of canonical number systems (CNS).

- Canonical number systems in integer rings $\mathbb{Z}_K$.
- Canonical number systems in orders $\mathbb{Z}[\alpha]$.
- If $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha$ then

$$\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/P(X)\mathbb{Z}[X].$$

Definition (Pethö, 1991)

We call $(P(X), \{0, \ldots, |P(0)| - 1\})$ a CNS if each element $z \in \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ admits a unique expansion of the form

$$z = d_0 + d_1 X + \cdots + d_\ell X^\ell \quad (0 \leq d_j < |P(0)|).$$
Possible generalizations of canonical number systems (CNS).

- Canonical number systems in integer rings $\mathbb{Z}_K$.
- Canonical number systems in orders $\mathbb{Z}[\alpha]$.
- If $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha$ then
  \[ Z[\alpha] \cong \mathbb{Z}[X]/P(X)\mathbb{Z}[X]. \]

**Definition (Pethö, 1991)**

We call $(P(X), \{0, \ldots, |P(0)| - 1\})$ a CNS if each element $z \in \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ admits a unique expansion of the form

\[ z = d_0 + d_1 X + \cdots + d_\ell X^\ell \quad (0 \leq d_j < |P(0)|). \]
Pethö’s general definition of CNS

Possible generalizations of **canonical number systems** (CNS).

- Canonical number systems in integer rings $\mathbb{Z}_K$.
- Canonical number systems in orders $\mathbb{Z}[\alpha]$.
- If $P(X) \in \mathbb{Z}[X]$ is the minimal polynomial of $\alpha$ then
  
  $$\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/P(X)\mathbb{Z}[X].$$

**Definition (Pethö, 1991)**

We call $(P(X), \{0, \ldots, |P(0)| - 1\})$ a CNS if each element $z \in \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ admits a unique expansion of the form

$$z = d_0 + d_1 X + \cdots + d_\ell X^\ell \quad (0 \leq d_j < |P(0)|).$$
Pethö’s general definition of CNS

Possible generalizations of canonical number systems (CNS).

- Canonical number systems in integer rings \( \mathbb{Z}_K \).
- Canonical number systems in orders \( \mathbb{Z}[\alpha] \).
- If \( P(X) \in \mathbb{Z}[X] \) is the minimal polynomial of \( \alpha \) then

\[
\mathbb{Z}[\alpha] \cong \mathbb{Z}[X]/P(X)\mathbb{Z}[X].
\]

Definition (Pethö, 1991)

We call \((P(X), \{0, \ldots, |P(0)| - 1\})\) a CNS if each element \( z \in \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) admits a unique expansion of the form

\[
z = d_0 + d_1 X + \cdots + d_\ell X^\ell \quad (0 \leq d_j < |P(0)|).
\]
Backward division

- **Base** \( P(X) = X^d + p_{d-1}X^{d-1} \cdots + p_0 \in \mathbb{Z}[X] \).
- **Digit set** \( D = \{0, 1, \ldots, p_0 - 1\} \).
- Each coset of \( \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) has a unique representant of the form \( A(x) = A_0 + A_1x + \cdots + A_{d-1}x^{d-1} \).
- The backward division mapping \( (p_d = 1) \):
  \[
  T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
  \]
  where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \).
- \( A(x) = (A_0 - qp_0) + xT_P(A) \) where \( A_0 - qp_0 \in D \).
- Iteration yields \( A(X) = d_0 + d_1x + \cdots + d_{\ell}x^\ell + T_P^\ell(A) \) with \( d_1, \ldots, d_{\ell} \in D \).
Backward division

- **Base** \( P(X) = X^d + p_{d-1}X^{d-1} \cdots + p_0 \in \mathbb{Z}[X] \).
- **Digit set** \( \mathcal{D} = \{0, 1, \ldots, p_0 - 1\} \).
- Each coset of \( \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) has a unique representant of the form \( A(x) = A_0 + A_1 x + \cdots + A_{d-1} x^{d-1} \).
- The **backward division mapping** \((p_d = 1)\):
  \[
  T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1}) X^i,
  \]
  where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \).
- \( A(x) = (A_0 - qp_0) + xT_P(A) \) where \( A_0 - qp_0 \in \mathcal{D} \).
- Iteration yields \( A(X) = d_0 + d_1 x + \cdots + d_\ell x^\ell + T_P^\ell(A) \) with \( d_1, \ldots, d_\ell \in \mathcal{D} \).
Backward division

- **Base** \( P(X) = X^d + p_{d-1}X^{d-1} \ldots + p_0 \in \mathbb{Z}[X] \).
- **Digit set** \( \mathcal{D} = \{0, 1, \ldots, p_0 - 1\} \).
- Each coset of \( \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) has a unique representant of the form \( A(x) = A_0 + A_1x + \cdots + A_{d-1}x^{d-1} \).
- The backward division mapping \((p_d = 1)\):

\[
T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
\]

where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \).
- \( A(x) = (A_0 - qp_0) + xT_P(A) \) where \( A_0 - qp_0 \in \mathcal{D} \).
- Iteration yields \( A(X) = d_0 + d_1x + \cdots + d_\ell x^\ell + T_P^\ell(A) \) with \( d_1, \ldots, d_\ell \in \mathcal{D} \).
Backward division

- **Base** \( P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{Z}[X] \).
- **Digit set** \( D = \{0, 1, \ldots, p_0 - 1\} \).
- Each coset of \( \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) has a unique representant of the form \( A(x) = A_0 + A_1x + \cdots + A_{d-1}x^{d-1} \).
- The **backward division mapping** \((p_d = 1)\):

\[
T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
\]

where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \).

- \( A(x) = (A_0 - qp_0) + xT_P(A) \) where \( A_0 - qp_0 \in D \).
- Iteration yields \( A(X) = d_0 + d_1x + \cdots + d_\ell x^\ell + T_P^\ell(A) \) with \( d_1, \ldots, d_\ell \in D \).
Backward division

- **Base** \( P(X) = X^d + p_{d-1}X^{d-1} \cdots + p_0 \in \mathbb{Z}[X] \).
- **Digit set** \( \mathcal{D} = \{0, 1, \ldots, p_0 - 1\} \).
- Each coset of \( \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \) has a unique representant of the form \( A(x) = A_0 + A_1x + \cdots + A_{d-1}x^{d-1} \).
- The **backward division mapping** \((p_d = 1)\):

\[
T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
\]

where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \).
- \( A(x) = (A_0 - qp_0) + xT_P(A) \) where \( A_0 - qp_0 \in \mathcal{D} \).
- Iteration yields \( A(X) = d_0 + d_1x + \cdots + d_\ell x^\ell + T_P^\ell(A) \) with \( d_1, \ldots, d_\ell \in \mathcal{D} \).
Backward division

- **Base** $P(X) = X^d + p_{d-1}X^{d-1} \cdots + p_0 \in \mathbb{Z}[X]$.
- **Digit set** $\mathcal{D} = \{0, 1, \ldots, p_0 - 1\}$.
- Each coset of $\mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ has a unique representant of the form $A(x) = A_0 + A_1x + \cdots + A_{d-1}x^{d-1}$.
- The backward division mapping ($p_d = 1$):
  \[
  T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
  \]
  where $A_d = 0$ and $q = \lfloor A_0/p_0 \rfloor$.
- $A(x) = (A_0 - qp_0) + xT_P(A)$ where $A_0 - qp_0 \in \mathcal{D}$.
- Iteration yields $A(X) = d_0 + d_1x + \cdots + d_\ell x^\ell + T_P^\ell(A)$ with $d_1, \ldots, d_\ell \in \mathcal{D}$. 
Relation between CNS and SRS

- $T_P$ can be regarded as a transformation on the coefficient vectors $(A_0, \ldots, A_{d-1})$.
- Via a base transformation $\Psi$ to the so-called Brunotte basis one can see that $T_P$ is conjugate to an SRS.
- In particular, let $r = (\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0})$. Then the following diagram is commutative:
The result

**Theorem**

Let \( P(X) := X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X] \) and set \( r := \left( \frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right) \). Then the following assertions hold.

- \((P, \{0, \ldots, |p_0| - 1\})\) is a CNS if and only if \( \tau_r \) has the finiteness property.
- \( T_P(A) \) is ultimately periodic for each \( A \in \mathbb{Z}[X]/P\mathbb{Z}[X] \) if and only if \( \tau_r(a) \) is ultimately periodic for each \( a \in \mathbb{Z}^d \).

**Example**

Since \((-1 + i, \{0, 1\})\) is a CNS we conclude that \( \tau_{(1/2,1)} \) is an SRS. Just note that \( x^2 + 2x + 2 \) is the minimal polynomial of \(-1 + i\).
The result

Theorem

Let \( P(X) := X^d + p_{d-1}X^{d-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X] \) and set
\[
\mathbf{r} := \left( \frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right).
\]
Then the following assertions hold.

- \((P, \{0, \ldots, |p_0| - 1\})\) is a CNS if and only if \( \tau_{\mathbf{r}} \) has the finiteness property.
- \( T_P(A) \) is ultimately periodic for each \( A \in \mathbb{Z}[X]/P\mathbb{Z}[X] \) if and only if \( \tau_{\mathbf{r}}(a) \) is ultimately periodic for each \( a \in \mathbb{Z}^d \).

Example

Since \((-1 + i, \{0, 1\})\) is a CNS we conclude that \( \tau_{(1/2, 1)} \) is an SRS. Just note that \( x^2 + 2x + 2 \) is the minimal polynomial of \(-1 + i\).
Some remarks CNS

- All quadratic polynomials that give rise to bases of CNS have been described by Kátaï, Kovács and Brunotte.
- Already for cubic polynomials the characterization of CNS bases is unknown.
- Knuth’s twin dragon can be generalized. With each CNS one can associate a self-affine tile.
- In view of the above theorem all these properties can be studied in the framework of shift radix systems.
Some remarks CNS

- All quadratic polynomials that give rise to bases of CNS have been described by Kátai, Kovács and Brunotte.
- Already for cubic polynomials the characterization of CNS bases is unknown.
- Knuth’s twin dragon can be generalized. With each CNS one can associate a self-affine tile.
- In view of the above theorem all these properties can be studied in the framework of shift radix systems.
Some remarks CNS

- All quadratic polynomials that give rise to bases of CNS have been described by Kátai, Kovács and Brunotte.
- Already for cubic polynomials the characterization of CNS bases is unknown.
- Knuth’s twin dragon can be generalized. With each CNS one can associate a self-affine tile.
- In view of the above theorem all these properties can be studied in the framework of shift radix systems.
Some remarks CNS

- All quadratic polynomials that give rise to bases of CNS have been described by Kátai, Kovács and Brunotte.
- Already for cubic polynomials the characterization of CNS bases is unknown.
- Knuth’s twin dragon can be generalized. With each CNS one can associate a self-affine tile.
- In view of the above theorem all these properties can be studied in the framework of shift radix systems.
Nonintegral bases: golden ratio

Let $\beta = \frac{1 + \sqrt{5}}{2}$ be the base.

We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 & \leq 3 < \beta^3 & \beta^2 & = -1 \beta^2 \\
\beta & \not\leq 3 - \beta^2 < \beta^2 & & = -0 \beta \\
1 & \not\leq 3 - \beta^2 < \beta & & = -01 \\
\beta^{-1} & \not\leq 3 - \beta^2 < 1 & & = -0 \beta^{-1} \\
\beta^{-2} & \leq 3 - \beta^2 < \beta & & = -1 \beta^{-2} \\
3 - \beta^2 - \beta^{-2} & = 0
\end{align*}
\]

$2.61803 \leq 3 < 4.23607$
Let $\beta = \frac{1+\sqrt{5}}{2}$ be the base.

We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 & \leq 3 < \beta^3 \\
\beta & \not\leq 3 - \beta^2 < \beta^2 \\
1 & \not\leq 3 - \beta^2 < \beta \\
\beta^{-1} & \not\leq 3 - \beta^2 < 1 \\
\beta^{-2} & \leq 3 - \beta^2 < \beta \\
3 - \beta^2 - \beta^{-2} & = 0
\end{align*}
\]

$1.61803 \not\leq 0.381966 < 2.61803$
Nonintegral bases: golden ratio

- Let $\beta = \frac{1+\sqrt{5}}{2}$ be the base.
- We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 & \leq 3 < \beta^3 \\
\beta & \leq 3 - \beta^2 < \beta^2 \\
1 & \leq 3 - \beta^2 < \beta \\
\beta^{-1} & \leq 3 - \beta^2 < 1 \\
\beta^{-2} & \leq 3 - \beta^2 < \beta \\
3 - \beta^2 - \beta^{-2} & = 0
\end{align*}
\]

\[
1 \leq 0.381966 < 1.61803
\]
Let $\beta = \frac{1 + \sqrt{5}}{2}$ be the base.

We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 &\leq 3 < \beta^3 & \Rightarrow & -1\beta^2 \\
\beta &\not\leq 3 - \beta^2 < \beta^2 & \Rightarrow & -0\beta \\
1 &\not\leq 3 - \beta^2 < \beta & \Rightarrow & -01 \\
\beta^{-1} &\not\leq 3 - \beta^2 < 1 & \Rightarrow & -0\beta^{-1} \\
\beta^{-2} &\leq 3 - \beta^2 < \beta & \Rightarrow & -1\beta^{-2} \\
3 - \beta^2 - \beta^{-2} &= 0
\end{align*}
\]

$0.61803 \not\leq 0.381966 < 1$
Let $\beta = \frac{1 + \sqrt{5}}{2}$ be the base.

We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 & \leq 3 < \beta^3 & \rightarrow & -1 \beta^2 \\
\beta & \nleq 3 - \beta^2 < \beta^2 & \rightarrow & -0 \beta \\
1 & \nleq 3 - \beta^2 < \beta & \rightarrow & -01 \\
\beta^{-1} & \nleq 3 - \beta^2 < 1 & \rightarrow & -0 \beta^{-1} \\
\beta^{-2} & \leq 3 - \beta^2 < \beta & \rightarrow & -1 \beta^{-2} \\
3 - \beta^2 - \beta^{-2} & = 0
\end{align*}
\]

$0.381966 \leq 0.381966 < 0.61803$
Let $\beta = \frac{1 + \sqrt{5}}{2}$ be the base.

We represent 3 w.r.t. base $\beta$ in greedy expansion:

\[
\begin{align*}
\beta^2 & \leq 3 < \beta^3 & & \text{\textbar} -1 \beta^2 \\
\beta & < 3 - \beta^2 < \beta^2 & & \text{\textbar} -0 \beta \\
1 & < 3 - \beta^2 < \beta & & \text{\textbar} -01 \\
\beta^{-1} & < 3 - \beta^2 < 1 & & \text{\textbar} -0 \beta^{-1} \\
\beta^{-2} & \leq 3 - \beta^2 < \beta & & \text{\textbar} -1 \beta^{-2} \\
3 - \beta^2 - \beta^{-2} & = 0
\end{align*}
\]

$3 = \beta^2 + \beta^{-2} = (100.01)_\beta$
Facts about the base $\beta = \frac{1+\sqrt{5}}{2}$

Let $\beta = \frac{1+\sqrt{5}}{2}$.

- $\beta^k \leq x < \beta^{k+1}$ implies that $x - \beta^k < \beta^{k-1}$.

- Thus 0,1 are the only digits needed and 11 cannot occur in a digit string.

- Each $x \in \mathbb{Z}[\beta] \cap [0, \infty)$ admits a finite greedy expansion. We say that $\frac{1+\sqrt{5}}{2}$ enjoys property (F).

We are interested in numbers $\beta$ admitting finite or at least periodic expansions.
Let $\beta = \frac{1+\sqrt{5}}{2}$.

- $\beta^k \leq x < \beta^{k+1}$ implies that $x - \beta^k < \beta^{k-1}$.
- Thus 0,1 are the only digits needed and 11 cannot occur in a digit string.
- Each $x \in \mathbb{Z}[\beta] \cap [0, \infty)$ admits a finite greedy expansion. We say that $\frac{1+\sqrt{5}}{2}$ enjoys property (F).

We are interested in numbers $\beta$ admitting finite or at least periodic expansions.
Facts about the base $\beta = \frac{1+\sqrt{5}}{2}$

Let $\beta = \frac{1+\sqrt{5}}{2}$.

- $\beta^k \leq x < \beta^{k+1}$ implies that $x - \beta^k < \beta^{k-1}$.
- Thus 0,1 are the only digits needed and 11 cannot occur in a digit string.
- Each $x \in \mathbb{Z}[\beta] \cap [0, \infty)$ admits a finite greedy expansion.
  
  We say that $\frac{1+\sqrt{5}}{2}$ enjoys property (F).

We are interested in numbers $\beta$ admitting finite or at least periodic expansions.
Facts about the base $\beta = \frac{1+\sqrt{5}}{2}$

Let $\beta = \frac{1+\sqrt{5}}{2}$.

- $\beta^k \leq x < \beta^{k+1}$ implies that $x - \beta^k < \beta^{k-1}$.

- Thus 0,1 are the only digits needed and 11 cannot occur in a digit string.

- Each $x \in \mathbb{Z}[\beta] \cap [0, \infty)$ admits a finite greedy expansion.
  We say that $\frac{1+\sqrt{5}}{2}$ enjoys property (F).

We are interested in numbers $\beta$ admitting finite or at least periodic expansions.
The general definition of greedy expansions

- $\beta > 1 \in \mathbb{R} \setminus \mathbb{Z}$ base
- $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ digits
- $\gamma \in [0, \infty)$ can be represented uniquely by
  \[
  \gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots
  \]
  with $a_i \in \mathcal{A}$ such that
  \[
  0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n
  \]
  holds for all $n \leq m$ (greedy expansion).
- $\beta$-expansions go back to Rényi, 1957 and Parry, 1960.
The general definition of greedy expansions

- $\beta > 1 \in \mathbb{R} \setminus \mathbb{Z}$ base
- $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ digits
- $\gamma \in [0, \infty)$ can be represented uniquely by

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$$

with $a_i \in \mathcal{A}$ such that

$$0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n$$

holds for all $n \leq m$ (greedy expansion).

$\beta$-expansions go back to Rényi, 1957 and Parry, 1960.
The general definition of greedy expansions

- $\beta > 1 \in \mathbb{R} \setminus \mathbb{Z}$ base
- $A = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ digits
- $\gamma \in [0, \infty)$ can be represented uniquely by

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$$

with $a_i \in A$ such that

$$0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n$$

holds for all $n \leq m$ (greedy expansion).

- $\beta$-expansions go back to Rényi, 1957 and Parry, 1960.
The general definition of greedy expansions

- \( \beta > 1 \in \mathbb{R} \setminus \mathbb{Z} \) (base)
- \( A = \{0, 1, \ldots, \lfloor \beta \rfloor \} \) (digits)
- \( \gamma \in [0, \infty) \) can be represented uniquely by
  \[
  \gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots
  \]
  with \( a_i \in A \) such that
  \[
  0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n
  \]
  holds for all \( n \leq m \) (greedy expansion).
- \( \beta \)-expansions go back to Rényi, 1957 and Parry, 1960.
beta-Transformation

For \( x \in [0, 1) \) this greedy expansion can be obtained by iterating the \( \beta \)-transformation

\[
T_\beta(x) = \beta x - \lfloor \beta x \rfloor.
\]

\[
x = \lfloor \beta x \rfloor \beta^{-1} + T_\beta(x) \beta^{-1} \\
= \lfloor \beta x \rfloor \beta^{-1} + \lfloor \beta T_\beta(x) \rfloor \beta^{-2} + T_\beta(x)^2 \beta^{-2} \\
= \sum_{j=1}^{\ell} \lfloor \beta T_\beta^j(x) \rfloor \beta^{-j} + T_\beta^\ell(x) \beta^{-\ell}
\]

If \( \beta \) is a Pisot number, the admissible digit strings form a shift of finite type or a sofic shift.
For $x \in [0, 1)$ this greedy expansion can be obtained by iterating the $\beta$-transformation

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$ 

$$x = \lfloor \beta x \rfloor \beta^{-1} + T_\beta(x) \beta^{-1}$$

$$= \lfloor \beta x \rfloor \beta^{-1} + \lfloor \beta T_\beta(x) \rfloor \beta^{-2} + T_\beta(x)^2 \beta^{-2}$$

$$= \sum_{j=1}^{\ell} \lfloor \beta T_\beta^{j-1}(x) \rfloor \beta^{-j} + T_\beta^{\ell}(x) \beta^{-\ell}$$

If $\beta$ is a Pisot number, the admissible digit strings form a shift of finite type or a sofic shift.
For $x \in [0, 1)$ this greedy expansion can be obtained by iterating the $\beta$-transformation

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

$$x = \lfloor \beta x \rfloor \beta^{-1} + T_\beta(x) \beta^{-1}$$

$$= \lfloor \beta x \rfloor \beta^{-1} + \lfloor \beta T_\beta(x) \rfloor \beta^{-2} + T_\beta(x)^2 \beta^{-2}$$

$$= \sum_{j=1}^{\ell} \lfloor \beta T_\beta^{j-1}(x) \rfloor \beta^{-j} + T_\beta^{\ell}(x) \beta^{-\ell}$$

If $\beta$ is a Pisot number, the admissible digit strings form a shift of finite type or a sofic shift.
Properties of beta-Expansions

- **Schmidt, 1980**: ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0, 1) \implies \beta$ is a Pisot or a Salem number.
- **Bertrand, 1977**: $\beta$ is a Pisot number $\implies$ each expansion of $\gamma \in \mathbb{Q}(\beta) \cap [0, \infty)$ is ultimately periodic.
- **Frougny and Solomyak, 1992**: $\beta > 1$ has property (F) if 
  \[ \text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty), \]
  (F) can hold only for Pisot numbers $\beta$. On the other hand, not all Pisot numbers have property (F).
- **Problem**: Characterize all Pisot numbers with property (F) (cf. Akiyama, Bórbely, Brunotte, Frougny, Hollander, Pethő, Solomyak, T.)
## Properties of beta-Expansions

- **Schmidt, 1980**: ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0, 1) \implies \beta$ is a Pisot or a Salem number.

- **Bertrand, 1977**: $\beta$ is a Pisot number $\implies$ each expansion of $\gamma \in \mathbb{Q}(\beta) \cap [0, \infty)$ is ultimately periodic.

- **Frougny and Solomyak, 1992**: $\beta > 1$ has property (F) if

  $$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty),$$

  (F) can hold only for Pisot numbers $\beta$. On the other hand, not all Pisot numbers have property (F).

- **Problem**: Characterize all Pisot numbers with property (F) (cf. Akiyama, Bőrbély, Brunotte, Frougny, Hollander, Pethő, Solomyak, T.)
Properties of beta-Expansions

- Schmidt, 1980: ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0, 1)$ $\implies \beta$ is a Pisot or a Salem number.
- Bertrand, 1977: $\beta$ is a Pisot number $\implies$ each expansion of $\gamma \in \mathbb{Q}(\beta) \cap [0, \infty)$ is ultimately periodic.
- Frougny and Solomyak, 1992: $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty),$$

(F) can hold only for Pisot numbers $\beta$. On the other hand, not all Pisot numbers have property (F).

- Problem: Characterize all Pisot numbers with property (F) (cf. Akiyama, Bórbely, Brunotte, Frougny, Hollander, Pethő, Solomyak, T.)
Properties of beta-Expansions

- **Schmidt, 1980**: ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0, 1)$ $\implies$ $\beta$ is a Pisot or a Salem number.

- **Bertrand, 1977**: $\beta$ is a Pisot number $\implies$ each expansion of $\gamma \in \mathbb{Q}(\beta) \cap [0, \infty)$ is ultimately periodic.

- **Frougny and Solomyak, 1992**: $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty),$$

(F) can hold only for Pisot numbers $\beta$. On the other hand, not all Pisot numbers have property (F).

- **Problem**: Characterize all Pisot numbers with property (F) (cf. Akiyama, Bőrbel, Brunotte, Frougny, Hollander, Pethő, Solomyak, T.)
Theorem

Let \( \beta > 1 \) be an algebraic integer with minimal polynomial \( X^d - a_1 X^{d-1} - \cdots - a_{d-1} X - a_d \). Set

\[
\begin{align*}
    r_1 & := 1, \\
    r_j & := a_j \beta^{-1} + a_{j+1} \beta^{-2} + \cdots + a_d \beta^{j-d-1} \quad (2 \leq j \leq d).
\end{align*}
\]

Then \( \beta \) has property (F) if and only if \( \tau_r \) with \( r = (r_d, \ldots, r_2) \) has the finiteness property.
Proof of the conjugacy

- \( x \in \mathbb{Z}[\beta] \).
- \( \mathbb{Z}[\beta] = \langle 1, \beta, \ldots, \beta^{d-1} \rangle_{\mathbb{Z}} = \langle r_1, \ldots, r_d \rangle_{\mathbb{Z}} \).
- Thus \( x = \sum_{i=1}^{d} z_{d-i} r_i \), where \( z_0, \ldots, z_{d-1} \in \mathbb{Z} \).
- The \( \beta \)-transform of \( x \) can be written as
  \[ T_{\beta}(x) = \sum_{i=1}^{d} z_{d+1-i} r_i \text{ with } z_d \text{ satisfying } (r_1 = 1) \]
  \[ 0 \leq z_d r_1 + z_{d-1} r_2 + \cdots + z_1 r_d < 1, \]
  i.e., \( z_d = -\lfloor z_{d-1} r_2 + \cdots + z_1 r_d \rfloor \).
- By the definition of \( \tau_r \) we see that
  \[ \tau_r(z_1, \ldots, z_{d-1}) = (z_2, \ldots, z_d). \]
Proof of the conjugacy

• \( x \in \mathbb{Z}[\beta] \).

• \( \mathbb{Z}[\beta] = \langle 1, \beta, \ldots, \beta^{d-1} \rangle_{\mathbb{Z}} = \langle r_1, \ldots, r_d \rangle_{\mathbb{Z}} \).

• Thus \( x = \sum_{i=1}^{d} z_{d-i} r_i \), where \( z_0, \ldots, z_{d-1} \in \mathbb{Z} \).

• The \( \beta \)-transform of \( x \) can be written as

\[
T_\beta(x) = \sum_{i=1}^{d} z_{d+1-i} r_i
\]

with \( z_d \) satisfying (\( r_1 = 1 \))

\[
0 \leq z_d r_1 + z_{d-1} r_2 + \cdots + z_1 r_d < 1,
\]

\( i.e., z_d = -\lfloor z_{d-1} r_2 + \cdots + z_1 r_d \rfloor \).

• By the definition of \( \tau_r \) we see that

\[
\tau_r(z_1, \ldots, z_{d-1}) = (z_2, \ldots, z_d).
\]
Proof of the conjugacy

- $x \in \mathbb{Z}[\beta]$.
- $\mathbb{Z}[\beta] = \langle 1, \beta, \ldots, \beta^{d-1} \rangle_\mathbb{Z} = \langle r_1, \ldots, r_d \rangle_\mathbb{Z}$.
- Thus $x = \sum_{i=1}^{d} z_{d-i} r_i$, where $z_0, \ldots, z_{d-1} \in \mathbb{Z}$.
- The $\beta$-transform of $x$ can be written as $T_\beta(x) = \sum_{i=1}^{d} z_{d+1-i} r_i$ with $z_d$ satisfying ($r_1 = 1$)
  
  $$0 \leq z_d r_1 + z_{d-1} r_2 + \cdots + z_1 r_d < 1,$$
  
  i.e., $z_d = -\lfloor z_{d-1} r_2 + \cdots + z_1 r_d \rfloor$.
- By the definition of $\tau_r$ we see that
  
  $$\tau_r(z_1, \ldots, z_{d-1}) = (z_2, \ldots, z_d).$$
Proof of the conjugacy

- $x \in \mathbb{Z}[\beta]$.  
- $\mathbb{Z}[\beta] = \langle 1, \beta, \ldots, \beta^{d-1} \rangle_{\mathbb{Z}} = \langle r_1, \ldots, r_d \rangle_{\mathbb{Z}}$.  
- Thus $x = \sum_{i=1}^{d} z_{d-i} r_i$, where $z_0, \ldots, z_{d-1} \in \mathbb{Z}$.  
- The $\beta$-transform of $x$ can be written as $T_\beta(x) = \sum_{i=1}^{d} z_{d+1-i} r_i$ with $z_d$ satisfying ($r_1 = 1$)
  $$0 \leq z_d r_1 + z_{d-1} r_2 + \cdots + z_1 r_d < 1,$$
  i.e., $z_d = -\lfloor z_{d-1} r_2 + \cdots + z_1 r_d \rfloor$.  
- By the definition of $\tau_r$ we see that
  $$\tau_r(z_1, \ldots, z_{d-1}) = (z_2, \ldots, z_d).$$
Proof of the conjugacy

- $x \in \mathbb{Z}[\beta]$.
- $\mathbb{Z}[\beta] = \langle 1, \beta, \ldots, \beta^{d-1} \rangle_\mathbb{Z} = \langle r_1, \ldots, r_d \rangle_\mathbb{Z}$.
- Thus $x = \sum_{i=1}^{d} z_{d-i} r_i$, where $z_0, \ldots, z_{d-1} \in \mathbb{Z}$.
- The $\beta$-transform of $x$ can be written as $T_\beta(x) = \sum_{i=1}^{d} z_{d+1-i} r_i$ with $z_d$ satisfying $(r_1 = 1)$
  \[ 0 \leq z_d r_1 + z_{d-1} r_2 + \cdots + z_1 r_d < 1, \]
  i.e., $z_d = -\lfloor z_{d-1} r_2 + \cdots + z_1 r_d \rfloor$.
- By the definition of $\tau_r$, we see that
  $\tau_r(z_1, \ldots, z_{d-1}) = (z_2, \ldots, z_d)$. 
The elements $r_2, \ldots, r_d$ can also be defined by

$$X^d - a_1 X^{d-1} - a_2 X^{d-2} - \ldots - a_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + r_3 X^{d-3} + \ldots + r_d).$$

In view of the conjugacy, ultimate periodicity of orbits of $T_\beta$ reflect in ultimate periodicity of orbits of $\tau_r$ and vice versa.

**Example**

Since $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$ the beta-transformation $T_{(1+\sqrt{5})/2}$ has property (F) if and only if $\tau_{(1-\sqrt{5})/2}$ enjoys the finiteness property.
Remarks

- The elements $r_2, \ldots, r_d$ can also be defined by
  \[
  X^d - a_1 X^{d-1} - a_2 X^{d-2} - \cdots - a_d =
  (X - \beta)(X^{d-1} + r_2 X^{d-2} + r_3 X^{d-3} + \cdots + r_d).
  \]
- In view of the conjugacy, ultimate periodicity of orbits of $T_\beta$
  reflect in ultimate periodicity of orbits of $\tau_r$ and vice versa.

Example

Since $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right) \left(x - \frac{1-\sqrt{5}}{2}\right)$ the
beta-transformation $T_{(1+\sqrt{5})/2}$ has property (F) if and only if
$\tau_{(1-\sqrt{5})/2}$ enjoys the finiteness property.
The elements $r_2, \ldots, r_d$ can also be defined by
\[
X^d - a_1 X^{d-1} - a_2 X^{d-2} - \cdots - a_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + r_3 X^{d-3} + \cdots + r_d).
\]

In view of the conjugacy, ultimate periodicity of orbits of $T_\beta$ reflect in ultimate periodicity of orbits of $\tau_r$ and vice versa.

**Example**

Since $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right) \left(x - \frac{1-\sqrt{5}}{2}\right)$ the beta-transformation $T_{(1+\sqrt{5})/2}$ has property (F) if and only if $\tau_{(1-\sqrt{5})/2}$ enjoys the finiteness property.
Periodicity and finiteness

The following sets are of importance.

- \( \mathcal{D}_d := \left\{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d, (\tau_r^k(a))_{k \geq 0} \text{ is ultimately periodic} \right\} \).

- \( \mathcal{D}_d^{(0)} := \left\{ r \in \mathbb{R}^d \mid \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0 \right\} \)

Obviously we have \( \mathcal{D}_d^{(0)} \subset \mathcal{D}_d \).

Describing these sets will imply finiteness and periodicity properties of CNS and beta-expansions.
Periodicity and finiteness

The following sets are of importance.

- \( \mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \mid \forall \mathbf{a} \in \mathbb{Z}^d, (\tau^k_r(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\} \).
- \( \mathcal{D}^{(0)}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \mid \forall \mathbf{a} \in \mathbb{Z}^d \exists k > 0 : \tau^k_r(\mathbf{a}) = 0 \right\} \)

Obviously we have \( \mathcal{D}^{(0)}_d \subset \mathcal{D}_d \).

Describing these sets will imply finiteness and periodicity properties of CNS and beta-expansions.
Periodicity and finiteness

The following sets are of importance.

- \( \mathcal{D}_d := \left\{ r \in \mathbb{R}^d | \forall a \in \mathbb{Z}^d, (\tau_r^k(a))_{k \geq 0} \text{ is ultimately periodic} \right\} \).
- \( \mathcal{D}_d^{(0)} := \left\{ r \in \mathbb{R}^d | \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0 \right\} \)

Obviously we have \( \mathcal{D}_d^{(0)} \subset \mathcal{D}_d \).

Describing these sets will imply finiteness and periodicity properties of CNS and beta-expansions.
**Theorem**

For $d = 1$ we have $D_1 = [-1, 1]$ and $D_1^{(0)} = [0, 1)$.

**Proof of the second assertion.**

- $\tau_r^k(a)$ can tend to 0 only if $|r| \leq 1$.
- For $-1 \leq r < 0$ we have $\tau_r(1) = -\lfloor r \rfloor = 1$.
- For $r = 1$ we get $\tau_r(1) = -1$ and, hence, $\tau_r^2(1) = 1$.
- For $0 \leq r < 1$ convergence to zero follows because $|\tau_r^2(x)| < |x|$ for $x \neq 0$.

**Corollary (Grünwald, 1889)**

$(q, \{0, 1, \ldots, |q| - 1\})$ is a CNS (i.e., admits finite expansions for all $x \in \mathbb{Z}$) if and only if $q \leq -2$. 


Dimension 1

Theorem

For $d = 1$ we have $D_1 = [-1, 1]$ and $D_1^{(0)} = [0, 1)$.

Proof of the second assertion.

- $\tau^k_r(a)$ can tend to 0 only if $|r| \leq 1$.
- For $-1 \leq r < 0$ we have $\tau_r(1) = -\lfloor r \rfloor = 1$.
- For $r = 1$ we get $\tau_r(1) = -1$ and, hence, $\tau^2_r(1) = 1$.
- For $0 \leq r < 1$ convergence to zero follows because $|\tau^2_r(x)| < |x|$ for $x \neq 0$.

Corollary (Grünwald, 1889)

$(q, \{0, 1, \ldots, |q| - 1\})$ is a CNS (i.e., admits finite expansions for all $x \in \mathbb{Z}$) if and only if $q \leq -2$. 
Dimension 1

**Theorem**

For $d = 1$ we have $D_1 = [-1, 1]$ and $D_1^{(0)} = [0, 1)$.

**Proof of the second assertion.**

- $\tau_r^k(a)$ can tend to 0 only if $|r| \leq 1$.
- For $-1 \leq r < 0$ we have $\tau_r(1) = -\lfloor r \rfloor = 1$.
- For $r = 1$ we get $\tau_r(1) = -1$ and, hence, $\tau_r^2(1) = 1$.
- For $0 \leq r < 1$ convergence to zero follows because $|\tau_r^2(x)| < |x|$ for $x \neq 0$.

**Corollary (Grünwald, 1889)**

$(q, \{0, 1, \ldots, |q| - 1\})$ is a CNS (i.e., admits finite expansions for all $x \in \mathbb{Z}$) if and only if $q \leq -2$. 
The characterization of $\mathcal{D}_2^{(0)}$: a picture

- $\mathcal{D}_2$ is (almost) the triangle.
- $\mathcal{D}_2^{(0)}$ is the yellow region.

Why???
The characterization of $\mathcal{D}_2^{(0)}$: a picture

- $\mathcal{D}_2$ is (almost) the triangle.
- $\mathcal{D}_2^{(0)}$ is the yellow region.

Why???
About $\mathcal{D}_2$

Recall: $\tau_r(a) = R(r) + (0, \ldots, 0, \{ra\})$

Here $\chi_r(x) = x^d + r_d x^{d-1} + \cdots + r_2 x + r_1$.

- $r \in \mathcal{D}_d$ if $\rho(R(r)) < 1$.
- $r \notin \mathcal{D}_d$ if $\rho(R(r)) > 1$.
- $\rho(R(r)) = 1$ is not so obvious

Questions:
- How does the region described by $\rho(R(r)) < 1$ look like?
- What about $\rho(R(r)) = 1$?
About \( D_2 \)

Recall: \( \tau_r(a) = R(r) + (0, \ldots, 0, \{ra\}) \)
Here \( \chi_r(x) = x^d + r_d x^{d-1} + \cdots + r_2 x + r_1 \).

- \( r \in D_d \) if \( \rho(R(r)) < 1 \).
- \( r \notin D_d \) if \( \rho(R(r)) > 1 \).
- \( \rho(R(r)) = 1 \) is not so obvious

Questions:
- How does the region described by \( \rho(R(r)) < 1 \) look like?
- What about \( \rho(R(r)) = 1 \)?
About $\mathcal{D}_2$

Recall: $\tau_r(a) = R(r) + (0, \ldots, 0, \{ra\})$
Here $\chi_r(x) = x^d + r_d x^{d-1} + \cdots + r_2 x + r_1$.

- $r \in \mathcal{D}_d$ if $\rho(R(r)) < 1$.
- $r \notin \mathcal{D}_d$ if $\rho(R(r)) > 1$.
- $\rho(R(r)) = 1$ is not so obvious

Questions:
- How does the region described by $\rho(R(r)) < 1$ look like?
- What about $\rho(R(r)) = 1$?
The Schur-Cohn Region

First we define the set

\[ \mathcal{E}_d := \{ (r_1, \ldots, r_d) \in \mathbb{R}^d : X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < 1 \} . \]

**Theorem (Schur, Takagi)**

The zeros of the polynomial

\[ g(X) = X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{R}[X] \]

are all contained in the unit disc iff the Hermitian form

\[
\sum_{\ell=0}^{n-1} |X_{\ell} + a_{n-1} X_{\ell+1} + \cdots + a_{\ell+1} X_{n-1}|^2
- \sum_{\ell=0}^{n-1} |a_0 X_{\ell} + a_1 X_{\ell+1} + \cdots + a_{n-1-\ell} X_{n-1}|^2
\]

is positive definite.
Determinant criterion

This is equivalent to

$$\delta_\nu > 0 \quad (0 \leq \nu \leq n - 1)$$

where $\delta_\nu$ is the determinant of

$$\begin{vmatrix}
1 & 0 & \ldots & 0 & a_0 & a_1 & \ldots & a_\nu \\
{a_{n-1}} & 1 & \ldots & 0 & 0 & a_0 & \ldots & a_{\nu-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
{a_{n-\nu}} & {a_{n-\nu+1}} & \ldots & 1 & 0 & 0 & \ldots & a_0 \\
a_0 & 0 & \ldots & 0 & 1 & a_{n-1} & \ldots & a_{n-\nu} \\
a_1 & a_0 & \ldots & 0 & 0 & 1 & \ldots & a_{n-\nu+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_\nu & a_{\nu-1} & \ldots & a_0 & 0 & 0 & \ldots & 1
\end{vmatrix}$$
$D_1$ and $D_2$

- $\mathcal{E}_1 = (-1, 1)$
- $\mathcal{E}_2 = \{(x, y) \in \mathbb{R}^2 : x < 1, -x - 1 < y < x + 1\}$
- $\mathcal{E}_d \subseteq D_d \subseteq \overline{\mathcal{E}_d}$
- $D_1 = [-1, 1]$
- $D_2$ (right side).
The “difficult” part of $D_2$

Here we have $r = (1, \lambda)$ with $|\lambda| \leq 2$

**Lemma**

Let $r$ be given as above. The sequence $(a_j)_{j \in \mathbb{N}}$ with $a_j = (a_j, a_{j+1})$ is an orbit of $\tau_r$ if and only if

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N}).$$

The question whether $\tau_{(1,\lambda)}$ has periodic orbits becomes now:

**Conjecture**

The sequence defined by

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N})$$

is periodic for all starting values $a_0, a_1$ and each $\lambda$ with $|\lambda| \leq 2$. 
The “difficult” part of $D_2$

Here we have $r = (1, \lambda)$ with $|\lambda| \leq 2$

Lemma

Let $r$ be given as above. The sequence $(a_j)_{j \in \mathbb{N}}$ with $a_j = (a_j, a_{j+1})$ is an orbit of $\tau_r$ if and only if

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N}).$$

The question whether $\tau_{(1,\lambda)}$ has periodic orbits becomes now:

Conjecture

The sequence defined by

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N})$$

is periodic for all starting values $a_0, a_1$ and each $\lambda$ with $|\lambda| \leq 2$. 
The “difficult” part of $D_2$

Here we have $r = (1, \lambda)$ with $|\lambda| \leq 2$

**Lemma**

Let $r$ be given as above. The sequence $(a_j)_{j \in \mathbb{N}}$ with $a_j = (a_j, a_{j+1})$ is an orbit of $\tau_r$ if and only if

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N}).$$

The question whether $\tau_{(1,\lambda)}$ has periodic orbits becomes now:

**Conjecture**

The sequence defined by

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N})$$

is periodic for all starting values $a_0, a_1$ and each $\lambda$ with $|\lambda| \leq 2$. 
The “difficult” part of $D_2$

Here we have $r = (1, \lambda)$ with $|\lambda| \leq 2$

**Lemma**

Let $r$ be given as above. The sequence $(a_j)_{j \in \mathbb{N}}$ with $a_j = (a_j, a_{j+1})$ is an orbit of $\tau_r$ if and only if

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N}).$$

The question whether $\tau_{(1, \lambda)}$ has periodic orbits becomes now:

**Conjecture**

The sequence defined by

$$0 \leq a_j + \lambda a_{j+1} + a_{j+2} < 1 \quad (j \in \mathbb{N})$$

is periodic for all starting values $a_0, a_1$ and each $\lambda$ with $|\lambda| \leq 2$. 
Partial answers

- If $\lambda \in \mathbb{Z}$ the conjecture is easily seen to be true.
- If $\lambda$ is a quadratic irrational number which makes the matrix $M_{(1,\lambda)}$ a rational rotation then the conjecture has been settled by Akiyama, Brunotte, Pethő and Steiner, 2008. These are the numbers $\frac{\pm 1 \pm \sqrt{5}}{2}$, $\pm \sqrt{2}$ and $\pm \sqrt{3}$. The proof is very difficult ($\pm \sqrt{3}$ is the worst case).
A small remark on the proof

A discontinuous non-ergodic piecewise affine map (Adler, Kitchens and Tessler, 2001).

- Let \( \lambda^2 = b\lambda + c \) be a quadratic irrational.
- Set \( x = \{\lambda a_{k-1}\} \) and \( y = \{\lambda a_k\} \).
- Then \( a_{k+1} = -a_{k-1} - \lambda a_k + y \).

Now \( \{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda'y\} \) (\( \lambda' \) is the conjugate of \( \lambda \)).

Thus we are left with studying the periodicity of

\[ T : [0, 1]^2 \rightarrow [0, 1]^2, \ (x, y) \mapsto (y, \{-x - \lambda'y\}). \]
A discontinuous non-ergodic piecewise affine map (Adler, Kitchens and Tessler, 2001).

- Let $\lambda^2 = b\lambda + c$ be a quadratic irrational.
- Set $x = \{\lambda a_{k-1}\}$ and $y = \{\lambda a_k\}$.
- Then $a_{k+1} = -a_{k-1} - \lambda a_k + y$.

Now \( \{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda'y\} \) (\( \lambda' \) is the conjugate of \( \lambda \)).

Thus we are left with studying the periodicity of

\[
T : [0, 1]^2 \to [0, 1]^2, \quad (x, y) \mapsto (y, \{-x - \lambda'y\}).
\]
A small remark on the proof

A discontinuous non-ergodic piecewise affine map (Adler, Kitchens and Tessler, 2001).

- Let $\lambda^2 = b\lambda + c$ be a quadratic irrational.
- Set $x = \{\lambda a_{k-1}\}$ and $y = \{\lambda a_k\}$.
- Then $a_{k+1} = -a_{k-1} - \lambda a_k + y$.

Now $\{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda'y\} \ (\lambda' \text{ is the conjugate of } \lambda)$. 

Thus we are left with studying the periodicity of

$$T : [0, 1]^2 \to [0, 1]^2, \ (x, y) \mapsto (y, \{-x - \lambda'y\}).$$
A small remark on the proof

A discontinuous non-ergodic piecewise affine map (Adler, Kitchens and Tessler, 2001).

- Let $\lambda^2 = b\lambda + c$ be a quadratic irrational.
- Set $x = \{\lambda a_{k-1}\}$ and $y = \{\lambda a_k\}$.
- Then $a_{k+1} = -a_{k-1} - \lambda a_k + y$.

Now $\{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda'y\}$ ($\lambda'$ is the conjugate of $\lambda$).

Thus we are left with studying the periodicity of

$$T : [0, 1]^2 \rightarrow [0, 1]^2, \ (x, y) \mapsto (y, \{-x - \lambda'y\}).$$
The set of aperiodic points of $T$
Constructing $D_d^{(0)}$ from $D_d$

Let $a_j := (a_{1+j}, \ldots, a_{d+j})$ ($0 \leq j \leq L - 1$) with $a_{L+1} = a_1, \ldots, a_{L+d} = a_d$ be vectors of $\mathbb{Z}^d$. We ask for which $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$ these vectors form a cycle

$$\pi : a_0 \xrightarrow{\tau_r} a_1 \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_r} a_{L-1} \xrightarrow{\tau_r} a_0$$

This is the case if and only if the inequalities

$$0 \leq r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (0 \leq j \leq L - 1) \quad (1)$$

We denote the polyhedron satisfying (1) by $\mathcal{P}(\pi)$. Since $r \in D_d^{(0)}$ if and only if $\tau_r$ has $0$ as its only period we conclude that

$$D_d^{(0)} = D_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi).$$
Constructing $\mathcal{D}_d^{(0)}$ from $\mathcal{D}_d$

Let $\mathbf{a}_j := (a_{1+j}, \ldots, a_{d+j}) \ (0 \leq j \leq L - 1)$ with $a_{L+1} = a_1, \ldots, a_{L+d} = a_d$ be vectors of $\mathbb{Z}^d$. We ask for which $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ these vectors form a cycle

$$\pi : \mathbf{a}_0 \xrightarrow{\tau_r} \mathbf{a}_1 \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_r} \mathbf{a}_{L-1} \xrightarrow{\tau_r} \mathbf{a}_0$$

This is the case if and only if the inequalities

$$0 \leq r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (0 \leq j \leq L - 1) \quad (1)$$

We denote the polyhedron satisfying (1) by $\mathcal{P}(\pi)$.

Since $\mathbf{r} \in \mathcal{D}_d^{(0)}$ if and only if $\tau_r$ has 0 as its only period we conclude that

$$\mathcal{D}_d^{(0)} = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi).$$
Constructing $\mathcal{D}^{(0)}_{d}$ from $\mathcal{D}_d$

Let $a_j := (a_{1+j}, \ldots, a_{d+j})$ ($0 \leq j \leq L - 1$) with $a_{L+1} = a_1, \ldots, a_{L+d} = a_d$ be vectors of $\mathbb{Z}^d$. We ask for which $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$ these vectors form a cycle

$$\pi : a_0 \xrightarrow{\tau_r} a_1 \xrightarrow{\tau_r} \cdots \xrightarrow{\tau_r} a_{L-1} \xrightarrow{\tau_r} a_0$$

This is the case if and only if the inequalities

$$0 \leq r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (0 \leq j \leq L - 1) \quad (1)$$

We denote the polyhedron satisfying (1) by $\mathcal{P}(\pi)$. Since $r \in \mathcal{D}^{(0)}_{d}$ if and only if $\tau_r$ has $0$ as its only period we conclude that

$$\mathcal{D}^{(0)}_{d} = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi).$$
Recall the picture of $D_2^{(0)}$

The “cut out” polygons come from the non-trivial periods.
A “naive” algorithm for describing $\mathcal{D}_d^{(0)}$

- Let $\mathbf{r}$ with $\rho(R(\mathbf{r})) < 1$ be given (i.e., $\mathbf{r} \in \text{int}(\mathcal{D}_d)$).
- $\tau_\mathbf{r}$ is a contraction. Thus there is a norm $\| \cdot \|$ and a constant $C$ such that
  \[ \forall \mathbf{a} \in \mathbb{Z}^d, \exists k \in \mathbb{N} : \| \tau_\mathbf{r}^k(\mathbf{a}) \| < C. \]
- Since the set $\{ \mathbf{a} \in \mathbb{Z}^d : \| \mathbf{a} \| < C \}$ is finite, we can check whether $\mathbf{r} \in \mathcal{D}_d$ in finitely many steps.
A “naive” algorithm for describing $\mathcal{D}_d^{(0)}$

- Let $\mathbf{r}$ with $\rho(R(\mathbf{r})) < 1$ be given (i.e., $\mathbf{r} \in \text{int}(\mathcal{D}_d)$).
- $\tau_\mathbf{r}$ is a contraction. Thus there is a norm $\| \cdot \|$ and a constant $C$ such that

$$\forall \mathbf{a} \in \mathbb{Z}^d, \exists k \in \mathbb{N} : \| \tau_\mathbf{r}^k(\mathbf{a}) \| < C.$$  

- Since the set $\{ \mathbf{a} \in \mathbb{Z}^d : \| \mathbf{a} \| < C \}$ is finite, we can check whether $\mathbf{r} \in \mathcal{D}_d$ in finitely many steps.
A “naive” algorithm for describing $\mathcal{D}_d^{(0)}$

- Let $\mathbf{r}$ with $\rho(R(\mathbf{r})) < 1$ be given (i.e., $\mathbf{r} \in \text{int}(\mathcal{D}_d)$).
- $\tau_\mathbf{r}$ is a contraction. Thus there is a norm $\| \cdot \|$ and a constant $C$ such that

$$\forall \mathbf{a} \in \mathbb{Z}^d, \exists k \in \mathbb{N} : \|\tau_\mathbf{r}^k(\mathbf{a})\| < C.$$ 

- Since the set $\{ \mathbf{a} \in \mathbb{Z}^d : \|\mathbf{a}\| < C \}$ is finite, we can check whether $\mathbf{r} \in \mathcal{D}_d$ in finitely many steps.
Brunotte’s algorithm

Theorem

Suppose that there exists a set \( \mathcal{V} \subset \mathbb{Z}^d \) satisfying

- \( \mathcal{V} \) contains \( 2d \) elements of the form \((0, \ldots, 0, \pm 1, 0, \ldots, 0)\).
- \( \tau_r(\mathcal{V}) \cup \tau_r^*(\mathcal{V}) \subset \mathcal{V} \), where \( \tau_r^*(x) = -\tau_r(-x) \).
- For each \( a \in \mathcal{V} \) there is some \( k > 0 \) such that \( \tau_r^k(a) = 0 \).

Then \( r \in D_d^{(0)} \). This criterion can be checked algorithmically.

This algorithm can be extended to give \( H \cap D_d^{(0)} \) for small compact sets \( H \subset \text{int}(D_d) \). This was used in order to draw the picture of \( D_2^{(0)} \).
Remarks

- The algorithms can only be used at some distance from \( \partial D_d \). If the spectral radius of \( R(r) \) tends to 1 their running time explodes.

- Near the boundary, sometimes other methods can be used, sometimes the description of \( D^{(0)}_d \) is still unknown (even for \( d = 2 \) near the left bounding line).

- Some regions of \( D^{(0)}_d \) can be characterized particularly easily.
The algorithms can only be used at some distance from $\partial D_d$. If the spectral radius of $R(r)$ tends to 1 their running time explodes.

Near the boundary, sometimes other methods can be used, sometimes the description of $D_d^{(0)}$ is still unknown (even for $d = 2$ near the left bounding line).

Some regions of $D_d^{(0)}$ can be characterized particularly easily.
The algorithms can only be used at some distance from $\partial D_d$. If the spectral radius of $R(r)$ tends to 1 their running time explodes.

Near the boundary, sometimes other methods can be used, sometimes the description of $D_d^{(0)}$ is still unknown (even for $d = 2$ near the left bounding line).

Some regions of $D_d^{(0)}$ can be characterized particularly easily.
Some characterization results I

**Theorem (Hollander; ABPT)**

If $\sum_{i=1}^{d} r_i < 1$ and $r_i \geq 0$ for $i \in \{1, \ldots, d\}$ then $r \in D_d^0$.

**Theorem (Akiyama und Rao; Scheicher und T.; ABPT)**

If $\sum_{i=1}^{d} |r_i| < 1$ and there exists exactly one index $k \in \{1, 2, \ldots, d\}$ such that $r_{d+1-k} < 0$. Then $r \in D_d^0$ if and only if

$$\sum_{1 \leq j \leq d/k} r_{d+1-kj} \geq 0.$$
Some characterization results I

Theorem (Hollander; ABPT)

If \( \sum_{i=1}^{d} r_i < 1 \) and \( r_i \geq 0 \) for \( i \in \{1, \ldots, d\} \) then \( r \in D_d^0 \).

Theorem (Akiyama und Rao; Scheicher und T.; ABPT)

If \( \sum_{i=1}^{d} |r_i| < 1 \) and there exists exactly one index \( k \in \{1, 2, \ldots, d\} \) such that \( r_{d+1-k} < 0 \). Then \( r \in D_d^0 \) if and only if

\[
\sum_{1 \leq j \leq d/k} r_{d+1-kj} \geq 0.
\]
Some characterization results II

Theorem (Kovacs; Frougny and Solomyak; ABPT)

If $0 \leq r_1 \leq r_2 \leq \cdots \leq r_d < 1$ then $r \in D_d^0$.

All these theorems can immediately be transformed in finiteness results for CNS and beta-expansions.
Characterization results for CNS

**Theorem (Kátai, Kovács; Brunotte)**

Let \( P(X) = X^2 + AX + B \in \mathbb{Z}[X] \). The pair \((P(X), \mathcal{N})\) is a CNS if and only if

\[-1 \leq A \leq B \quad \text{and} \quad B \geq 2.\]

**Theorem (Kovács, 1981)**

Let \( P(X) = \sum_{j=0}^{d} p_j X^j \). If \( p_0 \geq 2 \) and

\[ p_0 \geq p_1 \geq \ldots \geq p_{d-1} \geq 0 \]

then \((P, \mathcal{N})\) is a CNS.
Characterization results for CNS

**Theorem (Kátai, Kovács; Brunotte)**

Let \( P(X) = X^2 + AX + B \in \mathbb{Z}[X] \). The pair \((P(X), \mathcal{N})\) is a CNS if and only if

\[-1 \leq A \leq B \quad \text{and} \quad B \geq 2.\]

**Theorem (Kovács, 1981)**

Let \( P(X) = \sum_{j=0}^{d} p_j X^j \). If \( p_0 \geq 2 \) and

\[p_0 \geq p_1 \geq \ldots \leq p_{d-1} \geq 0\]

then \((P, \mathcal{N})\) is a CNS.