A Zoo of Continued Fraction Expansions
Part 2: $\alpha$-expansions

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June 7, 2011
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2 Hitoshi Nakada’s $\alpha$-expansions

3 The case $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$
   - Entropy of $T_\alpha$ ... recent results
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4 The case $\alpha \in \left[ \frac{5 + 2\sqrt{2}}{12 + 5\sqrt{2}}, \sqrt{2} - 1 \right)$

5 Rosen fractions, $\alpha$-Rosen fractions, and quilting
In 1981, Hitoshi Nakada introduced a family of continued fraction maps, and studied their natural extensions. These are the Nakada $\alpha$-expansions.

Let $\alpha \in [0, 1]$, the the continued fraction map $T_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ is defined by

$$T_\alpha(x) := \frac{\varepsilon(x)}{x} - \left\lfloor \frac{1}{x} \right\rfloor + 1 - \alpha, \quad x \neq 0,$$

and $T_\alpha(0) := 0$. Here $\varepsilon(x) = \text{sign}(x)$.

In case $\alpha = 1$, we have the regular (or: simple) continued fraction expansion (RCF), while the case $\alpha = \frac{1}{2}$ is the nearest integer continued fraction expansion (NICF). If $\alpha = 0$ we have the so-called by-excess continued fraction.
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If $T_{\alpha}^{n-1}(x) \neq 0$ for $n \in \mathbb{N}$, we define

$$
\varepsilon_n = \varepsilon_n(x) := \varepsilon \left( T_{\alpha}^{n-1}(x) \right), \quad \text{and} \quad a_n = a_n(x) := \left\lfloor \frac{1}{T_{\alpha}^{n-1}(x)} \right\rfloor + 1 - \alpha,
$$

From the definition of the map $T_{\alpha}$ one finds that

$$
x = \frac{\varepsilon_1}{a_1 + T_{\alpha}(x)} = \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + T_{\alpha}^2(x)}} = \cdots = \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots + \frac{\varepsilon_n}{a_n + T_{\alpha}^n(x)}}}.
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Taking “finite truncations” we get the $\alpha$-convergents of $x$;

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\frac{P_n}{Q_n} = \frac{\varepsilon_1}{a_1} + \frac{\varepsilon_2}{a_2} \cdots + \frac{\varepsilon_n}{a_n}, \quad n \geq 1.
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As in the previous talk one shows that

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x = \lim_{n \to \infty} \frac{P_n}{Q_n}.
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In the previous talk we mentioned insertions and singularizations.

If $A, B \in \mathbb{N}$, $\xi \in [0, 1)$, then

$$A + \frac{1}{1 + \frac{1}{B + \xi}} = A + 1 + \frac{-1}{B + 1 + \xi},$$

so singularizations can be used to ‘shorten’ the (regular) continued fraction expansions of numbers $x \in \mathbb{R}$. 
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so singularizations can be used to ‘shorten’ the (regular) continued fraction expansions of numbers \( x \in \mathbb{R} \).
Apart from singularizations there are insertions; let $A, B \in \mathbb{N}$, $B \geq 2$, and let $\xi \in [0, 1)$, then

$$A + \frac{1}{B + \xi} = A + 1 + \frac{-1}{1 + \frac{1}{B - 1 + \xi}}.$$ 

We have “inserted $-1/1$ between the digits $A$ and $B$.” The effect of an insertion is, that we have inserted a new convergent of the form

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$$

(a so-called mediant) in the sequence of RCF-convergents of $x$. 
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In the previous talk it was mentioned that for $\alpha \in [1/2, 1]$ the natural extension of $T_\alpha$ can be obtained as an $S$-expansion.

Due to this, the metric and arithmetic properties for these $\alpha$-expansions are easily derived from those of the RCF (and these are classical). Since this is old work details are skipped

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Note that due to the fact that for $\alpha \in [\frac{1}{2}, 1]$ the $\alpha$-expansion is an induced transformation, many properties can be obtained almost immediately. E.g., for $\alpha \in [\frac{1}{2}, g]$ the entropy is constant.

From the earlier work in 1999 of Marmi, Moussa and Cassa it follows that we can extend this interval to the interval for $\alpha \in [\sqrt{2} - 1, g]$.

However, in a paper in 2008 Laura Luzzi and Stefano Marmi used computer simulations to show that for $\alpha$ smaller than $\sqrt{2} - 1$ the entropy function has an erratic behavior; there seems to be intervals on which the entropy is constant, intervals on which it goes up, and intervals on which it goes down. Also the natural extension itself seems to have a strange form, with “holes” and other strange deformations. These ‘deformations’ we will try understand today.
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Also in a paper in 2008, Hitoshi Nakada and Rie Natsui were able to give an estimate of the decay rate of the entropy as $\alpha$ goes to zero. Then they show that there exist some decreasing sequences of intervals of $\alpha$, $(I_n)$, $(J_n)$, $(K_n)$ and $(L_n)$ such that $\frac{1}{n} \in I_n$, $I_{n+1} < J_n < L_n < I_n$, the entropy of $T_\alpha$ is increasing in $I_n$, decreasing on $K_n$ and constant on $J_n$ and $L_n$. (Here $I < J$ means $I \cap J = \emptyset$ and all elements of $I$ are smaller than all elements of $J$.)

Last year, Carlo Carminati, Stefano Marmi, Alessandro Profeti and Giulio Tiozzo further studied the entropy of $T_\alpha$. The behavior of entropy is known to be quite regular for parameters for which a matching condition on the orbits of the endpoints holds. They gave a detailed description of the set $M$ where this condition is met: it consists of a countable union of open intervals, corresponding to different combinatorial data, which appear to be arranged in a hierarchical structure.
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Their experimental data suggest that the complement of $\mathcal{M}$ has measure zero. Furthermore, the entropy on matching intervals is smooth. They can construct points outside of $\mathcal{M}$ on which it is not even locally monotone.

Tom Schmidt, Wolfgang Steiner and CK are finishing a paper in which we further give proofs of some of the properties observed. Actually, in this talk I will speak about the things which did not make it to our paper.

However, in order not to become very abstract we will approach $\alpha$-expansions for $\alpha$’s smaller than $\frac{1}{2}$ (and after that for $\alpha$’s smaller than $\sqrt{2} - 1$) in a hands-on manner.
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Entropy of $T_\alpha$ . . . recent results

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Let $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$, and let the nicf-expansion of $x \in [-\frac{1}{2}, \frac{1}{2})$ be given by

$$\begin{align*}
x &= \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots}} = [0; \varepsilon_1/a_1, \varepsilon_2/a_2, \ldots].
\end{align*}$$

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It is classical, that for the nicf one has that $a_n \geq 2$, and that $a_n + \varepsilon_{n+1} \geq 2$, for $n \geq 1$.

Nakada showed that the natural extension of the nearest integer continued fraction is the dynamical system $(\Omega_{\frac{1}{2}}, \bar{\mu}_{\frac{1}{2}}, T_{\frac{1}{2}})$,

where

$$\Omega_{\frac{1}{2}} = [-\frac{1}{2}, 0) \times [0, g^2] \cup [0, \frac{1}{2}) \times [0, g],$$
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\[ S(T(S')) \]

\[ g \]

\[ g^2 \]

\[ -\frac{1}{2} \]

\[ 0 \]

\[ \frac{1}{2} \]

\[ 1 \]
The natural extension for the NICF

The map $\mathcal{T}_{\frac{1}{2}} : \Omega_{\frac{1}{2}} \rightarrow \Omega_{\frac{1}{2}}$ is given by

$$\mathcal{T}_{\frac{1}{2}}(x, y) = \left( T_\alpha(x), \frac{1}{a(x) + \varepsilon(x)y} \right).$$

and $\bar{\mu}_\alpha$ is a $\mathcal{T}_{\frac{1}{2}}$-invariant probability measure on $\Omega_{\frac{1}{2}}$ with density

$$\frac{1}{\log G} \frac{1}{(1 + xy)^2}, \quad \text{for } (x, y) \in \Omega_{\frac{1}{2}}.$$
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Define for $n \geq 0$ the sequence of ‘futures’ $(t_n)_{n \geq 0}$ and the sequence of ‘pasts’ $(v_n)_{n \geq 0}$ of $x$, by $v_0 = 0$, and

$$(t_n, v_n) = T^n_{1/2}(x, 0), \quad n \in \mathbb{N} \cup \{0\}.$$ 

Note that for $n \geq 0$,

$$t_n = [0; \varepsilon_{n+1}/a_{n+1}, \varepsilon_{n+2}/a_{n+2}, \ldots], \quad \text{and} \quad v_n = [0; 1/a_n, \varepsilon_n/a_{n-1}, \ldots, \varepsilon_2/a_1],$$

and furthermore, if the nicf-convergents $p_n/q_n$ of $x$ are given by

$$p_n/q_n = [0; \varepsilon_1/a, \varepsilon_2/a_2, \ldots, \varepsilon_n/a_n], \quad n \geq 1,$$

with $\gcd(p_n, q_n) = 1$, then $q_n = a_nq_{n-1} + \varepsilon_nq_{n-2}$, and $v_n = q_{n-1}/q_n$. 
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with \( \gcd(p_n, q_n) = 1 \), then \( q_n = a_n q_{n-1} + \varepsilon_n q_{n-2} \), and \( v_n = q_{n-1}/q_n \).
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Now let $n \geq 0$ be such, that $t_n \in [\alpha, \frac{1}{2})$, so $t_n$ is “too big.”

Clearly, the nicf-expansion (1) of $x$ then satisfies

$$x = \frac{\varepsilon_1}{a_1 + \cdots + \frac{\varepsilon_n}{a_n + \frac{1}{2 + \frac{1}{a_{n+2} + \frac{\varepsilon_{n+3}}{a_{n+3} + \cdots}}}}}$$

$$= [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n, 1/2, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \ldots],$$

since $(t_n, v_n) \in [\alpha, \frac{1}{2}) \times [0, g]$ (implying that $a_{n+1} = 2$, and $t_{n+1} > 0$).
The natural extension for \( \alpha \in [\sqrt{2} - 1, \frac{1}{2}) \)

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= [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n, 1/2, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \ldots],
\]

since \((t_n, v_n) \in [\alpha, \frac{1}{2}) \times [0, g]\) (implying that \( a_{n+1} = 2 \), and \( t_{n+1} > 0 \)).
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Inserting $-1/1$ between $a_n$ and $a_{n+1} = 2$, yields a new continued fraction expansion of $x$,

\[
x = \frac{\varepsilon_1}{a_1 + \cdots + \frac{\varepsilon_n}{a_n + 1 + \frac{-1}{1 + \frac{1}{1 + \frac{1}{a_{n+2} + \frac{\varepsilon_{n+3}}{a_{n+3} + \cdots}}}}}} = [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/(a_n + 1), -1/1, 1/1, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \ldots].
\]
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Next we singularize in this new continued fraction expansion of $x$ the $n + 1$st partial quotient, arriving at

$$x = \frac{\varepsilon_1}{a_1 + \cdots + \frac{\varepsilon_n}{a_n + 1 + \frac{-1}{2 + \frac{-1}{a_{n+2} + 1 + \frac{\varepsilon_{n+3}}{a_{n+3} + \cdots}}}}}$$

which we denote by $x = [0; \varepsilon_1^*/d_1, \varepsilon_n^*/d_2, \ldots]$. 
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(2)

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\hspace{1cm} (2)
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Note that $\varepsilon_1^* = \varepsilon_1, \ldots, \varepsilon_n^* = \varepsilon_n, \varepsilon_{n+1}^* = -1 = \varepsilon_{n+2}^*$, and that $\varepsilon_{n+k}^* = \varepsilon_{n+k}$ for $k \geq 3$. Similarly, for the partial quotients we have that $d_1 = a_1, \ldots, d_{n-1} = a_{n-1}, d_n = a_n + 1, d_{n+1} = 2, d_{n+2} = a_{n+2} + 1$, and that $d_{n+k} = a_{n+k}$ for $k \geq 3$.

Setting

$$t_i^* = \frac{\varepsilon_{i+1}^*}{d_{i+1} + \cdots} = [0; \frac{\varepsilon_{i+1}^*}{d_{i+1}, \ldots}],$$

and

$$v_i^* = \frac{1}{d_i + \frac{\varepsilon_i^*}{d_{i-1} + \cdots + \frac{\varepsilon_2^*}{d_1}}} = [0; 1/d_i, \frac{\varepsilon_i^*}{d_{i-1}, \ldots, \varepsilon_2^*/d_1}],$$

we find that

$$(t_i^*, v_i^*) = (t_i, v_i), \text{ for } i = 1, \ldots, n - 1, \text{ and } i \geq n + 3,$$

and that $(t_n, v_n)$ and $(t_{n+1}, v_{n+1})$ got replaced by $(t_n^*, v_n^*)$ respectively $(t_{n+1}^*, v_{n+1}^*)$. 
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Note that $\varepsilon_1^* = \varepsilon_1, \ldots, \varepsilon_n^* = \varepsilon_n, \varepsilon_{n+1}^* = -1 = \varepsilon_{n+2}^*$, and that $\varepsilon_{n+k}^* = \varepsilon_{n+k}$ for $k \geq 3$. Similarly, for the partial quotients we have that $d_1 = a_1, \ldots, d_{n-1} = a_{n-1}$, $d_n = a_n + 1$, $d_{n+1} = 2$, $d_{n+2} = a_{n+2} + 1$, and that $d_{n+k} = a_{n+k}$ for $k \geq 3$.

Setting

$$t_i^* = \frac{\varepsilon_{i+1}^*}{d_{i+1} + \cdots} = [0; \varepsilon_{i+1}^*/d_{i+1}, \ldots],$$

and

$$v_i^* = \frac{1}{d_i + \frac{\varepsilon_i^*}{d_{i-1} + \cdots + \frac{\varepsilon_2^*}{d_1}}} = [0; 1/d_i, \varepsilon_i^*/d_{i-1}, \ldots, \varepsilon_2^*/d_1],$$

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$$(t_i^*, v_i^*) = (t_i, v_i), \quad \text{for } i = 1, \ldots, n - 1, \text{ and } i \geq n + 3,$$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Since

$$(t_n, v_n) \in R = [\alpha, \frac{1}{2}) \times [0, g],$$

we immediately have that

$$(t_{n+1}, v_{n+1}) \in T_{\frac{1}{2}}(R) = [0, \frac{1 - 2\alpha}{\alpha}) \times [g^2, \frac{1}{2}]$$

For the new continued fraction expansion (2) of $x$ these rectangles $R$ and $T_{\frac{1}{2}}(R)$ have been ‘vacated;’ see the figure on the next slide.

Where do $(t^*_n, v^*_n)$ and $(t^*_{n+1}, v^*_{n+1})$ ‘live’?
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for \( \alpha \in [\sqrt{2} - 1, \frac{1}{2}) \)

**Figure:** The regions \( \Omega_{\frac{1}{2}} \) and \( \Omega_{\alpha} \) for \( \alpha = 0.43 \). Here \( \ell_0 = \alpha - 1, \ell_1 = \frac{2\alpha - 1}{1 - \alpha}, r_o = \alpha, \) and \( r_1 = \frac{1 - \alpha}{\alpha} \).
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (2) we see that

\[
 t^*_n = \frac{-1}{2 + \frac{-1}{a_{n+2} + 1 + \cdots}},
\]

and since

\[
 t_n = \frac{1}{2 + \frac{1}{a_{n+2} + \cdots}} = 1 + \frac{-1}{1 + \frac{1}{a_{n+2} + \cdots}}
\]

\[
 = 1 + \frac{-1}{2 + \frac{-1}{a_{n+2} + 1 + \cdots}} = 1 + t^*_n,
\]

we obviously have that $t^*_n = t_n - 1$. 
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (2) we see that

$$t_n^* = \frac{-1}{2 + \frac{-1}{a_{n+2} + 1 + \ddots}}$$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Furthermore,

$$v_n^* = \frac{1}{a_n + \frac{\varepsilon_n}{a_{n-1} + \ddots}} = \frac{q_{n-1}}{(a_n + 1)q_{n-1} + \varepsilon_n q_{n-2}} = \frac{q_{n-1}}{q_n + q_{n-1}} = \frac{v_n}{1 + v_n}.$$ 

Thus,

$$(t_n, v_n) \in R = [\alpha, \sqrt{2} - 1) \times [0, g] \text{ if and only if } (t_n^*, v_n^*) \in [\alpha - 1, -\frac{1}{2}) \times [0, g^2].$$
The natural extension for \( \alpha \in [\sqrt{2} - 1, \frac{1}{2}) \)

Furthermore,

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\]
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (2) we also see that $t_{n+1}^* = [0; -1/(a_{n+2} + 1), \varepsilon_{n+3}/a_{n+3}, \ldots ]$, so (3) yields that

$$t_n^* = \frac{-1}{2 + t_{n+1}^*},$$

implying that

$$t_{n+1}^* = \frac{-1}{t_n^*} - 2.$$ (4)
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (2) we also see that $t^*_{n+1} = [0; -1/(a_{n+2} + 1), \varepsilon_{n+3}/a_{n+3}, \ldots]$, so (3) yields that

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(4)
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (2) we see that

$$v_{n+1}^* = \frac{1}{2 + \frac{-1}{a_n + 1 + \frac{\varepsilon_n}{a_{n-1} + \cdots}}} = [0; 1/2, -1/(a_n + 1), \varepsilon_n/a_{n-1}, \ldots, \varepsilon_2/a_1].$$

Since

$$v_n^* = \frac{1}{a_n + 1 + \frac{\varepsilon_n}{a_{n-1} + \cdots}},$$

we see that

$$v_{n+1}^* = \frac{1}{2 - v_n^*}.$$

(5)
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(5)
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

From (4) and (5) we now have that

$$(t_n^*, v_n^*) \in S(R) = [\alpha - 1, -\frac{1}{2}) \times [0, g^2]$$

if and only if

$$(t_{n+1}^*, v_{n+1}^*) \in S^2(R) = [\frac{2\alpha - 1}{1 - \alpha}, 0) \times [\frac{1}{2}, g];$$

It is easy to see that

$$\mathcal{T}_{\frac{1}{2}} \left([0, \frac{1-2\alpha}{\alpha}) \times [g^2, \frac{1}{2}]\right) = \mathcal{T}_\alpha \left([\frac{2\alpha - 1}{1 - \alpha}, 0) \times [\frac{1}{2}, g]\right),$$

which is equivalent to the earlier observation, that

$$(t_{n+3}, v_{n+3}) = (t_{n+3}^*, v_{n+3}^*).$$
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

So due to the insertions and singularizations we see that the map $S$ maps the region $R = [\alpha, \frac{1}{2}) \times [0, g]$ bijectively to $S(R) = [\alpha - 1, -\frac{1}{2}) \times [0, g^2]$, which in its turn is mapped by a map $S$ (which is in essence $T_\alpha$) to

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The regions $R$ and $T_{\frac{1}{2}}(R)$ are ‘vacated,’ and we saw that

$$T_{\frac{1}{2}}^2(R) = T_\alpha(S^2(R)),$$

i.e., the ‘hole’ $T_{\frac{1}{2}}(R)$ is filled up by the ‘new stuff’ $T_\alpha(S^2(R))$.

Note that all maps involved preserve the two-dimensional Gauss-measure.
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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2}]$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

Let’s look again at the natural extension we saw a few slides ago:
The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

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The natural extension for $\alpha \in [\sqrt{2} - 1, \frac{1}{2})$

In particular the case $\alpha = \sqrt{2} - 1 = 0.4142\ldots$ is rather interesting; in this case $\Omega_\alpha$ “consist” of two “islands:”
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This technique of “moving parts/rectangles of the natural extension around” has been applied (“blindly”) to other continued fraction algorithms as well, in particular to the Rosen fractions.

Starting from $\Omega_\alpha$ with $\alpha = \sqrt{2} - 1$, we could try to continue in the same way. Again we can use insertions and singularizations in order to find the “quilting maps,” (so not to “quilt” blindly), but clearly this is becoming a cumbersome business.

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The natural extension for $\alpha \in \left[ \frac{5 + 2\sqrt{2}}{12 + 5\sqrt{2}}, \sqrt{2} - 1 \right)$

Let $\alpha \in \left[ \frac{5 + 2\sqrt{2}}{12 + 5\sqrt{2}}, \sqrt{2} - 1 \right)$, and let $x \in [\sqrt{2} - 2, \sqrt{2} - 1)$ have as $\sqrt{2} - 1$-expansion the expansion given in (1).

The sequence of ‘futures’ $(t_n)_{n \geq 0}$ and ‘pasts’ $(v_n)_{n \geq 0}$ of $x$, for $n \geq 0$, are defined as in the nearest integer case, so

$$t_0 = x, \quad v_0 = 0,$$

and

$$(t_n, v_n) = T^n_{\sqrt{2} - 1}(x, 0), \quad n \in \mathbb{N} \cup \{0\}.$$
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The natural extension for \( \alpha \in \left[ \frac{5+2\sqrt{2}}{12+5\sqrt{2}}, \sqrt{2} - 1 \right) \)

The idea is now the same as in the previous case; Suppose that for some \( n \geq 0 \) we have that \( t_n \in [\alpha, \sqrt{2} - 1) \). A simple calculation shows that for the present choice of \( \alpha \) we have that

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\begin{align*}
  a_{n+1} &= 3, \quad \varepsilon_{n+1} = 1, \\
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i.e.,

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x = [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n, 1/3, -1/2, -1/a_{n+3}, \varepsilon_{n+4}/a_{n+4}, \ldots].
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Inserting \(-1/1\) between \( a_n \) and \( a_{n+1} = 3 \) now yields the following new continued fraction expansion of \( x \):

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Since there are no partial quotients equal to 1 in $\alpha$-expansions for $\alpha$ smaller than $g$, we insert again $-1/1$, now between the $n + 1$st partial quotient (which is 1) and the $n + 2$nd partial quotient (which is 2). This yields

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Note that the singularization we used here has a slightly different effect than the one we saw in the beginning of this talk; let $A, B \in \mathbb{N}$, $B \geq 2$, and let $\xi \in [0, 1)$, then

$$A + \frac{1}{1 + \frac{-1}{B + \xi}} = A + 1 + \frac{1}{B - 1 + \xi}.$$

Now the continued fraction expansion in (6) contains still a partial quotient equal to 1. Singularizing this partial quotient in the manner just described yields

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as a continued fraction of $x$.

Let us denote this continued fraction expansion of $x$ by $x = [0; \varepsilon_1^*/d_1, \varepsilon_n^*/d_2, \ldots]$, its ‘futures’ by $t_n^*$ and its ‘pasts’ by $v_n^*$. 
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As before we can now tie the $t_n$’s to $t_n^*$, et cetera, to find the “quilting maps.” I refrain from doing this, but let me show you in pictures how the quilting is taking place.
The natural extension for $\alpha \in \left[ \frac{5+2\sqrt{2}}{12+5\sqrt{2}}, \sqrt{2} - 1 \right)$

Figure: The unions of rectangles $R_1$ (shaded), $R_2$, $R_3$, and $R_4$. 
The natural extension for $\alpha \in \left[ \frac{5+2\sqrt{2}}{12+5\sqrt{2}}, \sqrt{2} - 1 \right)$

Note that the “islands” $R_{4,d}$ and $R_{4,u}$ are “sticking out” in the sense, that there are points in these sets with an $x$-coordinate larger than $\alpha$!

Tracking the removed parts (i.e., $R_{1,d}$ and $R_{1,u}$, and their images under the various maps) yields the following figures:
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**Figure:** The rectangles \( R_{2,d}, R_{2,u}, R_{2,d}^\dagger \) (light shade), and \( R_{2,u}^\dagger \) (dark shade).
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So we get the following natural extension (which is not $\Omega_\alpha$!! . . . we need to get rid of the “islands” $R_{4,d}$ and $R_{4,u}$).
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So we get the following natural extension (which is not \( \Omega_\alpha \))! \ldots we need to get rid of the “islands” \( R_{4,d} \) and \( R_{4,u} \).
Fortunately one has co-authors!

At this point one would mention the results from one’s own paper . . . which I will not do . . .
In 1954 David Rosen introduced an infinite family of continued fractions which generalize the nearest-integer continued fraction.

It is only very recently that the metrical properties of these so-called Rosen fractions have been investigated.

Let $q \in \mathbb{Z}, q \geq 3$ and $\lambda = \lambda_q = 2 \cos \frac{\pi}{q}$. The map $T_\lambda : \left[ -\frac{\lambda}{2}, \frac{\lambda}{2} \right) \rightarrow \left[ -\frac{\lambda}{2}, \frac{\lambda}{2} \right)$ is defined by

$$T_\lambda(x) := \left\lfloor \frac{1}{x} \right\rfloor - \lambda \left\lfloor \frac{1}{\lambda x} \right\rfloor + \frac{1}{2}, \quad \text{for } x \neq 0; \quad T_\lambda(0) := 0.$$  

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Repeatedly applying this operator to \( x \in \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \) yields the so-called Rosen expansion of \( x \).

Put \( d(x) = \left\lfloor \frac{1}{\lambda x} \right\rfloor + \frac{1}{2} \) and \( \varepsilon(x) = \text{sgn}(x) \). Furthermore, for \( n \geq 1 \) with \( T_{\lambda}^{n-1}(x) \neq 0 \) put

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\varepsilon_n(x) = \varepsilon_n = \varepsilon(T_{\lambda}^{n-1}(x)) \quad \text{and} \quad d_n(x) = d_n = d(T_{\lambda}^{n-1}(x)).
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This yields a continued fraction of the type

\[
x = \frac{\varepsilon_1}{d_1 \lambda + \frac{\varepsilon_2}{d_2 \lambda + \ldots}} = [\varepsilon_1 : d_1, \varepsilon_2 : d_2, \ldots],
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where \( \varepsilon \in \{\pm 1\} \) and \( d_i \in \mathbb{N}^+ \).

If \( q = 3 \) the Rosen fraction is the nearest integer continued fractions.
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In 1995 Hitoshi Nakada obtained the natural extension for Rosen fraction for even $q$'s. In 2000, Bob Burton, Tom Schmidt and CK found the natural extension for Rosen fractions for both odd and even $q$'s.

Essential are again the orbits of the end-points of the interval $[-\frac{\lambda}{2}, \frac{\lambda}{2})$ on which these Rosen fractions are ‘built.’
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Rosen fractions

$\Omega_q$ and $\mathcal{T}(\Omega_q)$ for $q = 8$:
In 2009 Karma Dajani, Wolfgang Steiner and CK introduced the so-called $\alpha$-Rosen fractions, and studied their metrical properties for special choices of $\alpha$.

These choices resemble Nakada’s $\alpha$-expansions; in fact for $q = 3$ these are Nakada’s $\alpha$-expansions.

To be more precise, let $q \in \mathbb{Z}$, $q \geq 3$, and $\lambda = \lambda_q = 2 \cos \frac{\pi}{q}$ (as before). Then we define for $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ the map $T_\alpha : [\lambda(\alpha - 1), \lambda \alpha] \to [\lambda(\alpha - 1), \lambda \alpha]$ by

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and \(T_\alpha(0) := 0\).
In order to have positive digits, we demand that $\alpha \leq 1/\lambda$. Setting $d(x) = \lceil \frac{1}{x\lambda} \rceil + 1 - \alpha$ (with $d(0) = \infty$), $\varepsilon(x) = \text{sgn}(x)$, and more generally

$$\varepsilon_n(x) = \varepsilon_n = \varepsilon(T_{\alpha}^{n-1}(x)) \quad \text{and} \quad d_n(x) = d_n = d(T_{\alpha}^{n-1}(x))$$

for $n \geq 1$, one obtains for $x \in I_{q,\alpha} := [\lambda(\alpha - 1), \alpha \lambda]$ an expression of the form

$$x = \frac{\varepsilon_1}{d_1 \lambda} + \frac{\varepsilon_2}{d_2 \lambda + \cdots + \frac{\varepsilon_n}{d_n \lambda + T_{\alpha}^n(x)}}$$

where $\varepsilon_i \in \{\pm 1, 0\}$ and $d_i \in \mathbb{N} \cup \{\infty\}$.

Note that the case $\alpha = 1/\lambda$ is the Rosen fraction equivalent of the classical regular continued fraction expansion (RCF) (in fact it is the RCF when $q = 3$).
In order to have positive digits, we demand that \( \alpha \leq \frac{1}{\lambda} \). Setting
\[
d(x) = \left\lfloor \frac{1}{x \lambda} \right\rfloor + 1 - \alpha \quad \text{(with } d(0) = \infty)\]
and more generally
\[
\varepsilon_n(x) = \varepsilon_n = \varepsilon \left( T_{\alpha}^{n-1}(x) \right) \quad \text{and} \quad d_n(x) = d_n = d \left( T_{\alpha}^{n-1}(x) \right)
\]
for \( n \geq 1 \), one obtains for \( x \in I_{q,\alpha} := [\lambda(\alpha - 1), \alpha \lambda] \) an expression of the form
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\(\alpha\)-Rosen fractions

In order to have positive digits, we demand that \(\alpha \leq 1/\lambda\). Setting
\[d(x) = \left\lfloor \left| \frac{1}{x\lambda} \right| + 1 - \alpha \right\rfloor \quad \text{(with } d(0) = \infty), \quad \varepsilon(x) = \text{sgn}(x), \text{ and more generally}
\]
\[\varepsilon_n(x) = \varepsilon_n = \varepsilon\left(T_{\alpha}^{n-1}(x)\right) \quad \text{and} \quad d_n(x) = d_n = d\left(T_{\alpha}^{n-1}(x)\right)
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\[x = \frac{\varepsilon_1}{d_1\lambda + \frac{\varepsilon_2}{d_2\lambda + \cdots + \frac{\varepsilon_n}{d_n\lambda + T_{\alpha}^{n}(x)}}},\]

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Note that the case \(\alpha = 1/\lambda\) is the Rosen fraction equivalent of the classical regular continued fraction expansion (RCF) (in fact it is the RCF when \(q = 3\)).
\( \Omega_q \) and \( T(\Omega_q) \) for \( q = 8 \):

**Figure 3.** The natural extension domain \( \Omega_\alpha \) (left) and its image under \( T_\alpha \) (right) of the \( \alpha \)-Rosen continued fraction \( (\overline{\delta_n} = -\delta_n) \); here \( q = 6 \), \( \alpha = 0.53 \), \( d_p(\ell_0) = 2 \), \( d_p(r_0) = 3 \).
α-Rosen fractions

Again – as in the “standard Rosen case” \((\alpha = \frac{1}{2})\) the behavior of the underlying dynamical system/natural extension is very different whether one considers the even or odd.

Essential in the construction of the natural extension (in the even case) is the following theorem.

**Theorem**

Let \(q = 2p, p \in \mathbb{N}, p \geq 2\), and let the sequences \((\ell_n)_{n \geq 0}\) and \((r_n)_{n \geq 0}\) be defined as before. If \(1/2 < \alpha < 1/\lambda\), then we have that

\[
\ell_0 < r_1 < \ell_1 < \cdots < r_{p-2} < \ell_{p-2} < -\delta_1 < r_{p-1} < 0 < \ell_{p-1} < r_0,
\]

\(d_p(r_0) = d_p(\ell_0) + 1\) and \(\ell_p = r_p\). If \(\alpha = 1/2\), then we have that

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\ell_0 = r_1 = \ell_1 < \cdots < r_{p-2} = \ell_{p-2} < -\delta_1 < r_{p-1} = \ell_{p-1} = 0 < r_0.
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\(\alpha\)-Rosen fractions

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If \(\alpha = 1/\lambda\), then we have that

\[
\ell_0 = r_1 < \ell_1 = r_2 < \cdots < \ell_{p-2} = r_{p-1} = -\delta_1 < 0 < r_0.
\]
Define the auxiliary sequence \((B_n)_{n \geq 0}\) by

\[
B_0 = 0, \quad B_1 = 1, \quad B_n = \lambda B_{n-1} - B_{n-2}, \quad \text{for } n = 2, 3, \ldots
\]

It is easy to see that \(B_n = \sin \frac{n\pi}{q} / \sin \frac{\pi}{q}\).

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We have the following theorem.
Theorem

Let $q = 2p$ with $p \geq 2$. Then the system of relations

$$
\begin{align*}
(\mathcal{R}_1) : & \quad H_1 = 1/(\lambda + H_{2p-1}) \\
(\mathcal{R}_2) : & \quad H_2 = 1/\lambda \\
(\mathcal{R}_n) : & \quad H_n = 1/(\lambda - H_{n-2}) \quad \text{for } n = 3, 4, \ldots, 2p - 1 \\
(\mathcal{R}_{2p}) : & \quad H_{2p-2} = \lambda/2 \\
(\mathcal{R}_{2p+1}) : & \quad H_{2p-3} + H_{2p-1} = \lambda
\end{align*}
$$

admits the (unique) solution

$$
H_{2n} = -\varphi_{p-n} = \frac{B_n}{B_{n+1}} = \frac{\sin \frac{n\pi}{2p}}{\sin \frac{(n+1)\pi}{2p}} \quad \text{for } n = 1, 2, \ldots, p - 1,
$$

$$
H_{2n-1} = \frac{B_{p-n} - B_{p+1-n}}{B_{p-1-n} - B_{p-n}} = \frac{\cos \frac{n\pi}{2p} - \cos \frac{(n-1)\pi}{2p}}{\cos \frac{(n+1)\pi}{2p} - \cos \frac{n\pi}{2p}} \quad \text{for } n = 1, 2, \ldots, p,
$$

in particular $H_{2p-1} = 1$. 
$\alpha$-Rosen fractions, even case

**Theorem**

*(Theorem continued)* Let $1/2 < \alpha < 1/\lambda$ and

$$\Omega_\alpha = \bigcup_{n=1}^{2p-1} J_n \times [0, H_n]$$

with $J_{2n-1} = [l_{n-1}, r_n)$, $J_{2n} = [r_n, l_n)$ for $n = 1, 2, \ldots, p - 1$, and $J_{2p-1} = [l_{p-1}, r_0)$. Then the map $T_\alpha : \Omega_\alpha \to \Omega_\alpha$ given by the standard natural extension map is bijective off of a set of Lebesgue measure zero.
Again the odd case is more “difficult,” and – as in the “standard Rosen case” ($\alpha = \frac{1}{2}$) consists of values $\alpha$ for which the entropy is constant, and $\alpha$ for which it is not.
In 2010, Tom Schmidt, Ionica Smeets and CK applied the idea of quilting to these \( \alpha \)-Rosen fractions. For \( \alpha \)’s smaller than \( \frac{1}{2} \) again ‘balconies’ can be observed:

\[ \alpha \text{-Rosen fractions via quilting} \]

We have the following theorem:

**Figure 2.** The added and deleted rectangles for \( \alpha = 0.48 \) in the natural extensions for \( q = 8 \).
In 2010, Tom Schmidt, Ionica Smeets and CK applied the idea of quilting to these $\alpha$-Rosen fractions. For $\alpha$’s smaller than $\frac{1}{2}$ again ‘balconies’ can be observed:

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**Figure 2.** The added and deleted rectangles for $\alpha = 0.48$ in the natural extensions for $q = 8$. 

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We have the following theorem:
Theorem

Fix $\in \mathbb{Z}$, $q \geq 4$, and $\lambda = \lambda_q = 2 \cos(\pi/q)$.

(i) Let

$$\alpha_0 = \begin{cases} \frac{\lambda^2 - 4 + \sqrt{(4 - \lambda^2)^2 + 4\lambda^2}}{2\lambda^2}, & \text{if } q \text{ is even} \\ \frac{\lambda - 2 + \sqrt{2\lambda^2 - 4\lambda + 4}}{\lambda^2}, & \text{if } q \text{ is odd}. \end{cases}$$

Then $(\alpha_0, 1/\lambda]$ is the largest interval (containing $1/2$) for which each natural extension of $T_\alpha$ is connected.

(ii) Furthermore, if

$$\omega_0 = \begin{cases} \frac{1}{\lambda}, & \text{if } q \text{ is even} \\ \frac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda + 4}}{2\lambda}, & \text{if } q \text{ is odd}. \end{cases}$$

Then the entropy of the $\alpha$-Rosen map for each $\alpha \in [\alpha_0, \omega_0]$ is equal to the entropy of the standard Rosen map.
As \( q \to \infty \) (so \( \lambda \to 2 \)) we see that \( \alpha_0 \) and \( \omega_0 \) converge (from below resp. from above) to 2:

Figure 1. Some values of \( \alpha_0 \) and of \( \omega_0 \). Even index on left, odd index on right.
For general $\alpha$-Rosen fractions (so for $q \geq 4$) the natural extensions for values of $\alpha$ smaller than $\alpha_0$ have not been investigated. However, the “fractal phenomenon” as observed by Luzzi and Marmi (and ‘explained’ in some earlier sheets) seem to appear also in general for $q$ even:

![Diagram showing change of topology at $\alpha = \alpha_0$](image)

**Figure 6.** Change of topology at $\alpha = \alpha_0$: Simulations of the natural extension for $q = 8$ with on the left $\alpha = \alpha_0 - 0.001$ and on the right $\alpha = \alpha_0 + 0.001$. 
... for $q$ odd:

Figure 8. Change of topology at $\alpha = \alpha_0$: Simulations of the natural extension for $q = 9$; on left $\alpha = \alpha_0 - 0.001$, on right $\alpha = \alpha_0 + 0.001$. 
Thank you for your attention!

Any Questions?
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