Multi-dimensional sets recognizable in all abstract numeration systems

Émilie Charlier, Anne Lacroix, Narad Rampersad

University of Liège

June 10th 2011
Plan

- Base $k$ numeration
- Multi-dimensional $k$-recognizable sets
- Abstract numeration systems
- Multi-dimensional $S$-recognizable sets
Plan

- Base $k$ numeration
- Multi-dimensional $k$-recognizable sets
- Abstract numeration systems
- Multi-dimensional $S$-recognizable sets
Base $k$ numeration

Let $k \geq 2$ be an integer.
$\Sigma_k = \{0, \ldots, k - 1\}$

$$n = \sum_{i=0}^{m} w_i k^i, \ w_m \neq 0$$

$rep_k(n) = w_m w_{m-1} \cdots w_0 \in \Sigma_k^*$

A set $X \subseteq \mathbb{N}$ is $k$-recognizable if the language $rep_k(X)$ is accepted by a finite automaton.
Base $k$ numeration

Proposition
If $X \subseteq \mathbb{N}$ is an ultimately periodic set, then $X$ is $k$-recognizable for all $k \geq 2$.

Theorem (Cobham, 1969)
Let $k, m \geq 2$ be two multiplicatively independent integers. A set $X \subseteq \mathbb{N}$ is both $k$-recognizable and $m$-recognizable if and only if $X$ is ultimately periodic.

Two integers $k$ and $m$ are *multiplicatively independent* if

$$k^r = m^s \Rightarrow r = s = 0$$
Corollary

A set $X \subseteq \mathbb{N}$ is $k$-recognizable for all $k \geq 2$ if and only if $X$ is ultimately periodic.
Corollary

A set $X \subseteq \mathbb{N}$ is $k$-recognizable for all $k \geq 2$ if and only if $X$ is ultimately periodic.

or equivalently

Corollary

A set $X \subseteq \mathbb{N}$ is $k$-recognizable for all $k \geq 2$ if and only if $X$ is 1-recognizable.

A set $X \subseteq \mathbb{N}$ is 1-recognizable if the language $\{a^n : n \in X\}$ of unary representations of $X$ is accepted by a finite automaton.
Plan

- Base $k$ numeration
- **Multi-dimensional** $k$-recognizable sets
- Abstract numeration systems
- Multi-dimensional $S$-recognizable sets
Multi-dimensional $k$-recognizable sets

**Padding function**

Let $w_1, \ldots, w_d$ be words over $\Sigma$, we define

$$(\cdot)^\# : (\Sigma^*)^d \rightarrow ((\Sigma \cup \{\#\})^d)^*$$

by

$$(w_1, \ldots, w_d)^\# = (w_1^\#^{m-|w_1|}, \ldots, w_d^\#^{m-|w_d|})$$

where $m = \max\{|w_1|, \ldots, |w_d|\}$

**Example**

$$(ab, a, cdb)^\# = (ab^\#, a^\#^\#, cdb) = (a, a, c)(b, \#, d)(\#, \#, b)$$
Multi-dimensional $k$-recognizable sets

Let $R \subseteq (\Sigma^*)^d$

$$R^\# = \{(w_1, \cdots, w_d)^\# \mid (w_1, \cdots, w_d) \in R\}$$

Let $k \geq 2$ be an integer and $X \subseteq \mathbb{N}^d$.

$$rep_k(X) = \{(rep_k(n_1), \ldots, rep_k(n_d)) \mid (n_1, \ldots, n_d) \in X\}$$

A set $X \subseteq \mathbb{N}^d$ is $k$-recognizable if the language $rep_k(X)^\#$ over $(\Sigma_k \cup \{\#\})^d$ is accepted by a finite automaton.
Multi-dimensional $k$-recognizable sets

Theorem (Cobham-Semenov, Semenov 1977)
Let $k, m$ be two multiplicatively independent integers. A subset $X$ of $\mathbb{N}^d$ is both $k$-recognizable and $m$-recognizable if and only if $X$ is semi-linear.

A set $X \subseteq \mathbb{N}^d$ is linear if there exist $v_0, v_1, \ldots, v_t \in \mathbb{N}^d$ such that

$$X = v_0 + \mathbb{N}v_1 + \mathbb{N}v_2 + \cdots + \mathbb{N}v_t.$$ 

A set $X \subseteq \mathbb{N}^d$ is semi-linear if it is a finite union of linear sets.
Multi-dimensional $k$-recognizable sets

Example

Figure: The set $X = \{(n, 2m) : n, m \in \mathbb{N}\} = \mathbb{N}(1, 0) + \mathbb{N}(0, 2)$
Multi-dimensional \( k \)-recognizable sets

**Corollary**

A set \( X \subseteq \mathbb{N}^d \) is \( k \)-recognizable for all \( k \geq 2 \) if and only if \( X \) is semi-linear.
Multi-dimensional $k$-recognizable sets

In the one dimensional case, we have the following equivalences

$X$ is ultimately periodic $\iff X$ is semi-linear $\iff X$ is 1-recognizable

$\Rightarrow$ The semi-linear sets are a good extension of ultimately periodic sets for the integer base numeration systems
Multi-dimensional $k$-recognizable sets

QUESTION : Is it also the case for the abstract numeration systems ?
Plan

- Base $k$ numeration
- Multi-dimensional $k$-recognizable sets
- Abstract numeration systems
- Multi-dimensional $S$-recognizable sets
Abstract numeration systems

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where $L$ is an infinite regular language over the totally ordered alphabet $(\Sigma, <)$

By enumerating words of $L$ in the radix order (induced by $<$), we define a one-to-one correspondence between $\mathbb{N}$ and $L$

$$\text{rep}_S : \mathbb{N} \rightarrow L : n \mapsto (n + 1)\text{th \ word \ of} \ L$$

$$\text{val}_S = \text{rep}_S^{-1} : L \rightarrow \mathbb{N}$$
Abstract numeration systems

Example

\[ S = (a^* b^*, \{a, b\}, a < b) \]

\[ L = a^* b^* \]

| \( \varepsilon \) | 0 |
| \( a \) | 1 |
| \( b \) | 2 |
| \( aa \) | 3 |
| \( ab \) | 4 |
| \( bb \) | 5 |
| \( aaa \) | 6 |
| \( aab \) | 7 |
| \( abb \) | 8 |
| \vdots | \vdots |

A set \( X \subseteq \mathbb{N} \) is \( S \)-recognizable if the language \( rep_S(X) \) is accepted by a finite automaton.
Abstract numeration systems

Remark

- The numeration system in base $k$ is an abstract numeration system built on the language

$$L = \{1, 2, \ldots, k - 1\}^\Sigma_k \cup \{\varepsilon\}$$
Abstract numeration systems

Remark

- The set \( \{n^2 : n \in \mathbb{N}\} \) is never \( k \)-recognizable but is \( S \)-recognizable for

\[
S = (a^* b^* \cup a^* c^*, \{a, b, c\}, a < b < c).
\]

<table>
<thead>
<tr>
<th>( L = a^* b^* \cup a^* c^* )</th>
<th>( \mathbb{N} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>0</td>
</tr>
<tr>
<td>( a )</td>
<td>1</td>
</tr>
<tr>
<td>( b )</td>
<td>2</td>
</tr>
<tr>
<td>( c )</td>
<td>3</td>
</tr>
<tr>
<td>( aa )</td>
<td>4</td>
</tr>
<tr>
<td>( ab )</td>
<td>5</td>
</tr>
<tr>
<td>( ac )</td>
<td>6</td>
</tr>
<tr>
<td>( bb )</td>
<td>7</td>
</tr>
<tr>
<td>( cc )</td>
<td>8</td>
</tr>
<tr>
<td>( aaa )</td>
<td>9</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
Abstract numeration systems

Theorem (Lecomte, Rigo, 2001)

*Ultimately periodic sets are $S$-recognizable for all ANS $S$ built on a regular language.*
Abstract numeration systems

Theorem (Lecomte, Rigo, 2001)

Ultimately periodic sets are $S$-recognizable for all ANS $S$ built on a regular language.

Corollary

A set $X \subseteq \mathbb{N}$ is $S$-recognizable for all ANS $S$ if and only if $X$ is ultimately periodic.

Corollary

A set $X \subseteq \mathbb{N}$ is $S$-recognizable for all ANS $S$ if and only if $X$ is 1-recognizable.
Plan

- Base $k$ numeration
- Multi-dimensional $k$-recognizable sets
- Abstract numeration systems
- Multi-dimensional $S$-recognizable sets
Multi-dimensional $S$-recognizable sets

Let $S = (L, \Sigma, \prec)$ an abstract numeration system and $X \subseteq \mathbb{N}^d$.

$$\text{rep}_S(X) = \{(\text{rep}_S(n_1), \ldots, \text{rep}_S(n_d)) \mid (n_1, \ldots, n_d) \in X\}$$

A set $X \subseteq \mathbb{N}^d$ is $S$-recognizable if the language $\text{rep}_S(X)^\#$ over $(\Sigma \cup \{\#\})^d$ is accepted by a finite automaton.

$X$ is 1-recognizable if it is $S$-recognizable for the ANS $S$ built on $a^*$.
Multi-dimensional $S$-recognizable sets

Are the semi-linear sets a good extension of ultimately periodic sets for abstract numeration systems?
Multi-dimensional $S$-recognizable sets

Are the semi-linear sets a good extension of ultimately periodic sets for abstract numeration systems?

NO because in multi-dimensional case, semi-linear $\neq$ 1-recognizable
Multi-dimensional $S$-recognizable sets

Are the semi-linear sets a good extension of ultimately periodic sets for abstract numeration systems?

**NO** because in multi-dimensional case, semi-linear $\neq$ 1-recognizable

**Example**
The semi-linear set $X = \{(n, 2n)|n \in \mathbb{N}\}$ is not 1-recognizable.
Indeed, the unary representation of $X$

$$R^\# = \{(a^n \#, a^{2n})|n \in \mathbb{N}\}$$

is not regular.
Multi-dimensional $S$-recognizable sets

Theorem (Charlier, L, Rampersad 2010)

A subset $X$ of $\mathbb{N}^d$ is $S$-recognizable for all ANS $S$ if and only if $X$ is 1-recognizable.

$\Rightarrow$ 1-recognizable sets are a good generalization of ultimately periodic sets for abstract numeration systems.
Example

Consider the set

\[ X = \{(2n, 3m + 1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\} \cup \{(n, 2m) : n, m \in \mathbb{N} \text{ and } n < 2m\}. \]
Example

It is clear that \( X_1 = \{(2n, 3m + 1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\} \) is 1-recognizable...

Figure: Automaton accepting unary representation of \( X_1 \)
Example

... and that $X_2 = \{(n, 2m) : n, m \in \mathbb{N} \text{ and } n < 2m\}$ is 1-recognizable.

Figure: Automaton accepting unary representation of $X_2$
Example

Let $S$ be an ANS. The set $X_1 = \{(2n, 3m + 1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\}$ is $S$-recognizable.

Indeed, the sets $\{2n : n \in \mathbb{N}\}$ and $\{3m + 1 : m \in \mathbb{N}\}$ are $S$-recognizable by the result of Lecomte and Rigo.
Example

Let $S$ be an ANS. The set $X_1 = \{(2n, 3m + 1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\}$ is $S$-recognizable.

Indeed, the sets $\{2n : n \in \mathbb{N}\}$ and $\{3m + 1 : m \in \mathbb{N}\}$ are $S$-recognizable by the result of Lecomte and Rigo.

So the set $X' = \{(2n, 3m + 1) : n, m \in \mathbb{N}\}$ is $S$-recognizable, i.e. the language

$$A = \{(\text{rep}_S(2n), \text{rep}_S(3m + 1))\# : n, m \in \mathbb{N}\}$$

is accepted by a finite automaton.
Example

Let $S$ be an ANS. The set $X_1 = \{(2n, 3m + 1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\}$ is $S$-recognizable.

Indeed, the sets $\{2n : n \in \mathbb{N}\}$ and $\{3m + 1 : m \in \mathbb{N}\}$ are $S$-recognizable by the result of Lecomte and Rigo.

So the set $X' = \{(2n, 3m + 1) : n, m \in \mathbb{N}\}$ is $S$-recognizable, i.e. the language

$$A = \{(rep_S(2n), rep_S(3m + 1))\# : n, m \in \mathbb{N}\}$$

is accepted by a finite automaton

Moreover, $B = \{(x, y)\# : x, y \in L, x \geq y\}$ is also accepted by a finite automaton
Example

Let $S$ be an ANS. The set $X_1 = \{(2n, 3m+1) : n, m \in \mathbb{N} \text{ and } 2n \geq 3m + 1\}$ is $S$-recognizable.

Indeed, the sets $\{2n : n \in \mathbb{N}\}$ and $\{3m+1 : m \in \mathbb{N}\}$ are $S$-recognizable by the result of Lecomte and Rigo.

So the set $X' = \{(2n, 3m+1) : n, m \in \mathbb{N}\}$ is $S$-recognizable, i.e. the language

$$A = \{(rep_S(2n), rep_S(3m+1))\# : n, m \in \mathbb{N}\}$$

is accepted by a finite automaton.

Moreover, $B = \{(x, y)\# : x, y \in L, x \geq y\}$ is also accepted by a finite automaton.

Then, $rep_S(X_1)\# = A \cap B$ is accepted by a finite automaton and $X_1$ is $S$-recognizable.
We can construct in the same way an automaton accepting the $S$-representations of

$$X_2 = \{(n, 2m) : n, m \in \mathbb{N} \text{ and } n < 2m\}$$

Since the union of two regular languages is regular, we have that $X = X_1 \cup X_2$ is $S$-recognizable.
Multi-dimensional $S$-recognizable sets

Let $A$ be a non-empty subset of $\{1, \ldots, d\}$. Define the subalphabet

$$\Sigma_A = \{x \in (\Sigma \cup \{\#\})^d : \text{the } i\text{-th component of } x \text{ is } \# \text{ exactly when } i \notin A\}.$$ 

Example

Let $\Sigma = \{a\}$ and $d = 4$.
If $A = \{1, 2, 3, 4\}$, then $\Sigma_A = \{(a, a, a, a)\}$.
If $A = \{1, 3\}$, then $\Sigma_A = \{(a, \#, a, \#)\}$. 
Multi-dimensional $S$-recognizable sets

Let $A$ be a non-empty subset of $\{1, \ldots, d\}$. Define the subalphabet

$$\Sigma_A = \{x \in (\Sigma \cup \{\#\})^d : \text{the } i\text{-th component of } x \text{ is } \# \text{ exactly when } i \notin A\}.$$ 

**Theorem (Decomposition, Eilenberg, Elgot, Shepherdson 1969)**

Let $R \subseteq (\Sigma^*)^d$. The language $R^\# \subseteq ((\Sigma \cup \{\#\})^d)^*$ is regular if and only if it is a finite union of languages of the form

$$R_0 \cdots R_t, \quad t \in \mathbb{N},$$

where each factor $R_i \subseteq (\Sigma_{A_i})^*$ is regular and $A_t \subseteq \cdots \subseteq A_0 \subseteq \{1, \ldots, d\}$. 

Multi-dimensional $S$-recognizable sets

Example
Let $X = \{(5n, 5n + 4m + 1, 5n + 4m + 3, 5n) : n, m, \in \mathbb{N}\}$.

The unary representation of $X$ is

$$R\# = ((a, a, a, a)^5)^*((\#, a, a, \#)^4)^*(\#, a, a, \#)(\#, \#, a, \#)^2.$$ 

Since $R\#$ is regular the set $X$ is 1-recognizable.
Multi-dimensional $S$-recognizable sets

Example
Let $X = \{(5n, 5n + 4m + 1, 5n + 4m + 3, 5n) : n, m, \in \mathbb{N}\}$. The unary representation of $X$ is

$$R^\# = ((a, a, a, a)^5)^*((\#, a, a, \#)^4)^*(\#, a, a, \#)(\#, \#, a, \#)^2.$$ 

Since $R^\#$ is regular the set $X$ is 1-recognizable.

The set $X$ can be written as

$$X = \{5(n, n, n, n)+4(0, m, m, 0)+(0, 1, 1, 0)+(0, 0, 2, 0) : n, m \in \mathbb{N}\},$$
Conclusion

In the multi-dimensional case, the sets that are $S$-recognizable for all abstract numeration systems $S$ are exactly the 1-recognizable sets.
Thank you...