Primitive matrices over polynomial semirings

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Theorem (Perron - Frobenius)

Let \( A \in \mathbb{R}^{r \times r} \) have only nonnegative entries. Then \( A \) is primitive if and only if \( A \) is irreducible and aperiodic.
Characterization of real primitive matrices

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Our aim:

Generalization of this result to matrices with polynomial entries
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$R$ a unital commutative semiring of characteristic 0
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$\mathcal{P}_+$ a nonempty additively and multiplicatively closed subset of $R$ with $0 \notin \mathcal{P}_+$ ('positive' elements)
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$\mathcal{P} = \mathcal{P}_+ \cup \{0\}$ (’nonnegative’ elements)
We call the matrix $A \in P^{r \times r}$

- $P_+$-primitive if there is some $m \in \mathbb{N}_{>0}$ such that $A^m \in P_+^{r \times r}$. The least such $m$ is called the (primitive) exponent of $A$ w.r.t. $P_+$ and denoted by $\gamma_{P_+}(A)$. 

Primitivity, irreducibility and aperiodicity

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- $\mathcal{P}_+$-irreducible if for every $i,j \in [r] = \{1, \ldots, r\}$ there is some $m \in \mathbb{N}$ such that $(A^m)_{ij} \in \mathcal{P}_+$.
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- $\mathcal{P}_+$-aperiodic if the greatest common divisor of the set

$$\{\text{per}_1(A, \mathcal{P}_+), \ldots, \text{per}_r(A, \mathcal{P}_+)\}$$

equals 1 where we denote by $\text{per}_i(A, \mathcal{P}_+)$ the greatest common divisor of the set

$$\{n \in \mathbb{N}_{>0} : (A^n)_{ii} \in \mathcal{P}_+\}$$

if this set is nonvoid, and $\text{per}_i(A, \mathcal{P}_+) = \infty$, otherwise.
The easy case $r = 1$
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**Theorem**

Let

$$f = \sum_{k=0}^{\deg(f)} \kappa_k(f)X^k \in \mathcal{P}[X]$$

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$$\kappa_0(f), \kappa_1(f), \kappa_{\deg(f)-1}(f) \in \mathcal{P}_+.$$
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**Notation:**

$$\text{slc}(f) = \begin{cases} \kappa_{\deg(f)-1}(f), & \text{if } \deg(f) > 0 \\ \text{slc}(f) = 0, & \text{otherwise.} \end{cases}$$
Theorem

Let $A \in \mathcal{P}[X]^{r \times r}$ and assume that $A_{ij}$ is $\mathcal{P}_+[X]$-primitive for all $i, j \in [r]$. Then $A$ is $\mathcal{P}_+[X]$-primitive.
Matrices with primitive entries

**Theorem**

Let $A \in \mathcal{P}[X]^{r \times r}$ and assume that $A_{ij}$ is $\mathcal{P}_+[X]$-primitive for all $i, j \in [r]$. Then $A$ is $\mathcal{P}_+[X]$-primitive. Moreover, if

$$\deg(A) = \max \{ \deg A_{ij} : i, j \in [r] \} \leq 3$$

we have $A \in \mathcal{P}_+[X]^{r \times r}$,
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we have $A \in \mathcal{P}_+[X]^{r \times r}$, and otherwise

$$\gamma_{\mathcal{P}_+[X]}(A) \leq 2 \deg(A) - 3,$$

and this constant is best possible.
Generalization of an example of Perron

Let $f_1, \ldots, f_r \in \mathcal{P}[X]$ and

$$A = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & f_1 \\
1 & 0 & \cdots & \cdots & 0 & f_2 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & f_{r-1} \\
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\end{pmatrix}.$$
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If $f_1$ and $f_r$ are $\mathcal{P}_+[X]$-primitive and $f_2, \ldots, f_{r-1}$ are either $\mathcal{P}_+[X]$-primitive or zero then $A$ is $\mathcal{P}_+[X]$-primitive.
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$$\gamma_{\mathcal{P}_+[X]}(A) \leq \max \left\{ 2 \max \{ \deg(f_1), \ldots, \deg(f_r) \} - 1, 4r^2(r - 1) + 1 \right\}.$$
Modification of the conditions on the entries of $A$

We call $A$

- strongly $\mathcal{P}_+[X]$-irreducible if for all $i,j \in [r]$ there is some $n \in \mathbb{N}$ such that the following three properties are satisfied:

1. $\kappa_0(A^n)_{ij} \in \mathcal{P}_+$
2. $\deg(A^n)_{ij} \geq \min\{1, \deg(A)\}$
3. $\deg(A^n)_{ij} > 0 \implies \kappa_1(A^n)_{ij} \in \mathcal{P}_+$. 

Note:

- strong $\mathcal{P}_+[X]$-irreducibility = $\mathcal{P}_+[X]$-irreducibility
- $\deg(A) = 0 \implies$ strong $\mathcal{P}_+[X]$-irreducibility (strong $\mathcal{P}_+[X]$-aperiodicity, resp.) coincides with $\mathcal{P}_+[X]$-irreducibility (strong $\mathcal{P}_+[X]$-aperiodicity, resp.)
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- strongly $\mathcal{P}_+[X]$-aperiodic if the greatest common divisor of the set $\{\text{sper}_1(A), \ldots, \text{sper}_r(A)\}$ equals 1, where for $i \in [r]$ we denote by $\text{sper}_i(A)$ the greatest common divisor of the set of all positive integers which satisfy the three properties of strong irreducibility if this set is non-void, and $\text{sper}_i(A) = \infty$, otherwise.
Modification of the conditions on the entries of \( A \)

We call \( A \)

\begin{itemize}
  \item strongly \( \mathcal{P}_+[X] \)-irreducible if for all \( i, j \in [r] \) there is some \( n \in \mathbb{N} \) such that the following three properties are satisfied:
    \begin{enumerate}
      \item \( \kappa_0(A^n)_{ij} \in \mathcal{P}_+ \)
      \item \( \deg(A^n)_{ij} \geq \min \{1, \deg(A)\} \)
      \item \( \deg(A^n)_{ij} > 0 \implies \kappa_1(A^n)_{ij} \in \mathcal{P}_+ \).
    \end{enumerate}
  
  \item strongly \( \mathcal{P}_+[X] \)-aperiodic if the greatest common divisor of the set \( \{\text{sper}_1(A), \ldots, \text{sper}_r(A)\} \) equals 1, where for \( i \in [r] \) we denote by \( \text{sper}_i(A) \) the greatest common divisor of the set of all positive integers which satisfy the three properties of strong irreducibility if this set is non-void, and \( \text{sper}_i(A) = \infty \), otherwise.
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Note:

- strong $\mathcal{P}_+[X]$-irreducibility $\implies \mathcal{P}_+[X]$-irreducibility
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Our first primitivity criterion

Theorem

The following statements are equivalent for $A \in \mathcal{P}[X]^{r \times r}$.

(i) $A$ is $\mathcal{P}_+[X]$-primitive.
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(i) $A$ is $\mathcal{P}_+[X]$-primitive.

(ii) $A$ is strongly $\mathcal{P}_+[X]$-irreducible and strongly $\mathcal{P}_+[X]$-aperiodic.
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Theorem

The following statements are equivalent for $A \in \mathcal{P}[X]^{r \times r}$.

(i) $A$ is $\mathcal{P}_+[X]$-primitive.

(ii) $A$ is strongly $\mathcal{P}_+[X]$-irreducible and strongly $\mathcal{P}_+[X]$-aperiodic and there exists some positive integer $n$ such that

$$\slc(A^n)_{ij} \in \mathcal{P}_+.$$ 

for all $i, j \in [r]$. 
An example of strong irreducibility and aperiodicity

Let

\[ A = \begin{pmatrix} X & 1 + X^2 \\ 1 + X^3 & 1 + X^4 \end{pmatrix} \in \mathbb{R}_{\geq 0}[X]^{2 \times 2}. \]
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We have

\[
A^2 = \begin{pmatrix} 1 + 2X^2 + X^3 + X^5 & 1 + X + X^2 + X^3 + X^4 + X^6 \\ 1 + X + X^3 + 2X^4 + X^7 & 2 + X^2 + X^3 + 2X^4 + X^5 + X^8 \end{pmatrix},
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But we have

\[
\text{slc}(A_n)_{11} = 0 \quad (n \in \mathbb{N}),
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thus \(A\) is not \(R_0[X]\)-primitive.
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and for \( n \geq 3 \) and \( i, j \in \{1, 2\} \) we find

\[ \kappa_0(A^n)_{ij} > 0, \quad \kappa_1(A^n)_{ij} > 0, \quad \deg (A^n)_{ij} \geq 9. \]
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Thus \( A \) is strongly \( \mathbb{R}_{>0}[X] \)-irreducible and strongly \( \mathbb{R}_{>0}[X] \)-aperiodic.
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Thus \( A \) is strongly \( \mathbb{R}_{>0}[X] \)-irreducible and strongly \( \mathbb{R}_{>0}[X] \)-aperiodic. But we have
\[ \text{slc}(A^n)_{11} = 0 \quad (n \in \mathbb{N}), \]

thus \( A \) is not \( \mathbb{R}_{>0}[X] \)-primitive.
Controlling the \slc\-coefficients of the powers of $A$

For $f \in R[X]$ we set

$$\delta(f) = \begin{cases} 1, & \text{if } \slc(f) \in \mathcal{P}_+, \\ 0, & \text{otherwise.} \end{cases}$$
Controlling the slc-coefficients of the powers of $A$

For $f \in R[X]$ we set

$$\delta(f) = \begin{cases} 1, & \text{if } \text{slc}(f) \in P_+, \\ 0, & \text{otherwise.} \end{cases}$$

We define a map

$$s : \begin{cases} R[X] \to (\mathbb{N} \cup \{-\infty\}) \times \{0, 1\} \\ f \mapsto (\deg(f), \delta(f)) \end{cases}.$$
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Our aim: Endow $s$ with more structure!
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$$\mathcal{D} = ((\mathbb{N} \cup \{-\infty\}) \times \{0, 1\}) \setminus \{(-\infty, 1), (0, 1)\}.$$
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we introduce two binary operations:

$$(n, a) \oplus (m, b) = (\max \{ n, m \}, \delta_+(n, a, m, b))$$
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$$(n, a) \oplus (m, b) = (\max\{n, m\}, \delta_+(n, a, m, b))$$

and

$$(n, a) \otimes (m, b) = (n + m, \delta_\times(n, a, m, b)) .$$
Completion of the definitions of the binary operations on $\mathcal{D}$

Definition of the function $\delta_+ : \mathcal{D} \times \mathcal{D} \rightarrow \{0, 1\}$ :
Completion of the definitions of the binary operations on $\mathcal{D}$

Definition of the function $\delta_+ : \mathcal{D} \times \mathcal{D} \rightarrow \{0, 1\}$:
$\delta_+ (n, a, m, b) = 1$ if one of the following three conditions is satisfied:

(i) $\max \{|m - n|, a, b\} = 1$,
(ii) $n > m + 1$ and $a = 1$,
(iii) $m > n + 1$ and $b = 1$,

otherwise $\delta_+ (n, a, m, b) = 0$. 

$(\mathcal{D}, \oplus, \otimes)$ is a commutative dioid with neutral elements $\epsilon = (-\infty, 0)$ and $e = (0, 0)$, respectively.
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Definition of the function $\delta_\times : \mathcal{D} \times \mathcal{D} \longrightarrow \{0, 1\}$:

$\delta_\times(n, a, m, b) = \max\{a, b\}$ if $n, m \in \mathbb{N}$, and $\delta_\times(n, a, m, b) = 0$, otherwise.

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$(\mathcal{D}, \oplus, \otimes)$ is a commutative dioid with neutral elements $\varepsilon = (-\infty, 0)$ and $e = (0, 0)$, respectively.
Relation between $R[X]$ and $\mathcal{D}$

$\mathcal{D}$ is a commutative unital semiring of characteristic 0 with idempotent sum (i.e., $\alpha \oplus \alpha = \alpha$ for every $\alpha \in \mathcal{D}$).

The mapping $s: \{R[X] \rightarrow \mathcal{D} \}$ is a ring homomorphism which we extend to $s: R[X] \times R[X] \rightarrow \mathcal{D} \times \mathcal{D}$.

The subset $Q = \{ (n, 1) \in \mathcal{D} : n \in \mathbb{N} > 0 \}$ is additively and multiplicatively closed and $\varepsilon = (\infty, 0) \in Q$. 
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The mapping

$$s : \begin{cases} R[X] & \rightarrow \mathcal{D} \\ f & \mapsto (\deg(f), \delta(f)) \end{cases}$$

is a ring homomorphism.
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Relation between $R[X]$ and $D$

$D$ is a commutative unital semiring of characteristic 0 with idempotent sum (i.e., $\alpha \oplus \alpha = \alpha$ for every $\alpha \in D$).

The mapping 

$$s : \begin{cases} R[X] & \longrightarrow D \\ f & \mapsto (\deg(f), \delta(f)) \end{cases}$$

is a ring homomorphism which we extend to

$$s : R[X]^{r \times r} \longrightarrow D^{r \times r}.$$ 

The subset 

$$Q = \{(n, 1) \in D : n \in \mathbb{N}_{>0}\}$$

is additively and multiplicatively closed and 

$$\varepsilon = (-\infty, 0) \notin Q.$$
Our second primitivity criterion

For $A \in R[X]^{r \times r}$ we denote by $\kappa_k(A)$ the matrix in $R^{r \times r}$ with entries

$$(\kappa_k(A))_{ij} = \kappa_k(A_{ij}) \quad (k \in \mathbb{N}, \; i, j \in [r]).$$
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$$
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$$

**Theorem**

Let $A \in \mathcal{P}[X]^{r \times r}$ have positive degree. Then $A$ is $\mathcal{P}_+[X]$-primitive if and only if the following conditions hold:
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**Theorem**

Let $A \in \mathcal{P}[X]^{r \times r}$ have positive degree. Then $A$ is $\mathcal{P}_+[X]$-primitive if and only if the following conditions hold:

(i) $\kappa_0(A)$ is $\mathcal{P}_+$-primitive.

(ii) $\kappa_1(A)$ is nonzero.
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(iii) $s(A)$ is $Q$-irreducible and $Q$-aperiodic.
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Our aim: Enhance condition (iii)!
Almost linear periodic sequences in monoids

Definition
The sequence \( a^* = (a_n)_{n \in \mathbb{N}} \) of elements of the monoid \( (\mathcal{M}, \circ) \) is called almost linear periodic if there are \( N \in \mathbb{N}, \ p \in \mathbb{N}_{>0} \) and \( q \in \mathcal{M} \) such that for every \( n > N \)

\[
(*) \quad a_{n+p} = a_n \circ q^p.
\]

The smallest number \( N \) with the property that there are \( p \in \mathbb{N}_{>0} \) and \( q \in \mathcal{M} \) such that \((*)\) holds for every \( n > N \) is called the linear defect of \( a^* \), and we write \( N = \text{ldef} a^* \).

The minimal number \( p \in \mathbb{N}_{>0} \) such that there is some \( q \in \mathcal{M} \) such that \((*)\) holds for every \( n > \text{ldef} a^* \) is called the linear period of \( a^* \), and we write \( p = \text{lper} a^* \).

An element \( q \) with \((*)\) is called a linear factor of \( a^* \).

In case \( q \) is unique we write \( q = \text{lfac} a^* \).
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Example
$a^* = (n+1, 1)_{n \in \mathbb{N}} \in \mathcal{D}$ is almost linear periodic with $\text{ldef } a^* = 0$ and $\text{lper } a^* = 1$, and $(1, 0)$ and $(1, 1)$ are linear factors of $a^*$. 
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The smallest number \(N\) with the property that there are \(p \in \mathbb{N}_{>0}\) and \(q \in M\) such that \((*)\) holds for every \(n > N\) is called the linear defect of \(a^*\), and we write \(N = \text{ldef } a^*\). The minimal number \(p \in \mathbb{N}_{>0}\) such that there is some \(q \in M\) such that \((*)\) holds for every \(n > \text{ldef } a^*\) is called the linear period of \(a^*\), and we write \(p = \text{lper } a^*\). An element \(q\) with \((*)\) is called a linear factor of \(a^*\). In case \(q\) is unique we write \(q = \text{lfac } a^*\).

Example
\(a^* = (n + 1, 1)_{n \in \mathbb{N}} \in D^\mathbb{N}\) is almost linear periodic with \(\text{ldef } a^* = 0\) and \(\text{lper } a^* = 1\), and \((1, 0)\) and \((1, 1)\) are linear factors of \(a^*\).
Let $B$ be an $(r \times r)$-matrix over a commutative dioid. We call $B$ almost linear periodic if for all $i, j \in [r]$ the sequence $(B^{n})_{ij} = ((B_{n})_{ij})_{n \in \mathbb{N}}$ is almost linear periodic.

The numbers $l_{\text{per}}(B) = \text{lcm}\{l_{\text{per}}((B^{n})_{ij}) : i, j \in [r]\}$ and $l_{\text{def}}(B^{\star}) = \max\{l_{\text{def}}((B^{n})_{ij}) : i, j \in [r]\}$ are called the linear period and the linear defect, respectively, of $B$.

A matrix $Q$ given by linear factors $Q_{ij}$ of the sequences $((B_{n})_{ij})_{n \in \mathbb{N}} (i, j \in [r])$ is called a linear factor matrix of $B$.

If there is a unique linear factor matrix we write $\text{lfac}(B) = \text{lfac}((B_{n})_{n \in \mathbb{N}})$.
Generalization of notions of M. Gavalec (2000)

Let $B$ be an $(r \times r)$-matrix over a commutative dioid. We call $B$ almost linear periodic.
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Irreducible matrices over commutative semirings

Let $A \in R^{r \times r}$.

- The digraph $G(A)$ is the weighted digraph $([r], E, v)$ with vertex set $[r]$, arc set

$$E = \{(i, j) \in [r]^2 : A_{ij} \neq 0\}$$

and weight function $v : E \to R \setminus \{0\}$ with

$$v(i, j) = A_{ij}$$

for all $(i, j) \in E$. Note: $P^+\text{-irreducibility}$ implies irreducibility, but the converse does not hold (e.g., take $(0, 0) \in D$ and $P^+ = Q$).
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Note: $\mathcal{P}_+$-irreducibility implies irreducibility, but the converse does not hold (e.g., take $(0, 0) \in \mathcal{D}$ and $\mathcal{P}_+ = \mathbb{Q}$).
Theorem (M. Gavalec (2000))

Let $M$ be a max-plus algebra generated by a divisible abelian linearly ordered group in additive notation (for instance, $\mathbb{R}_{\text{max}} = (\mathbb{R} \cup \{-\infty\}, \max, +)$).

Let $B \in M_{r \times r}$ be irreducible. Then the following statements hold.

- $B$ is almost linear periodic.
- $(\lfac(B))_{ij} = \lambda(B)$ for each $i, j \in [r]$ where $\lambda(B)$ denotes the maximal cycle mean weight of $G(B)$.
- $\lper(B)$ can be computed in $O(r^3)$ time.
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A remark on max-plus algebras

Definition
Let \((G, +, \leq)\) be an abelian linearly ordered divisible group

Example \((\mathbb{R}, +, \leq)\) generates \(\mathbb{R}_{\text{max}} = (\mathbb{R} \cup \{-\infty\}, \max, +)\).

Remark \(D\) cannot be embedded into a max-plus algebra because otherwise \((1, 1) = (1, 0) \oplus (0, 0) = \max\{1, 0\} \in \{1, 0\}\): Contradiction!
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Contradiction!
Splitting up a matrix over $\mathcal{D}$

Let $B \in \mathcal{D}^{r \times r}$. 

\[ F \in \mathbb{R}^{r \times r} \text{ max} = \text{matrix of first components of } B, \quad i.e., \quad F_{ij} = (B_{ij})_{1} \quad (i, j \in \left[\right]r) \]

\[ S \in \{0, 1\}^{r \times r} = \text{matrix of second components of } B_{n}, \quad i.e., \quad (S_{n})_{ij} = ((B_{n})_{ij})_{2} \quad (i, j \in \left[\right]r) \]

Note:
\[ F_{n} = (B_{n})_{1} \quad (n \in \mathbb{N}) \text{ where addition and multiplication on the left hand side is performed in } \mathbb{R}^{\text{max}}. \]

Every cycle of maximum mean weight in $G(B)$ is a cycle of maximum mean weight in $G(F)$ (some more technical effort is needed to define the mean weight of a cycle in $G(B)$).
Splitting up a matrix over $\mathcal{D}$

Let $B \in \mathcal{D}^{r \times r}$.

- $F \in \mathbb{R}^{r \times r}_{\text{max}}$ = matrix of first components of $B$, i.e.,
  \[ F_{ij} = (B_{ij})_1 \quad (i, j \in [r]). \]
Splitting up a matrix over $\mathcal{D}$

Let $B \in \mathcal{D}^{r \times r}$.

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- $S_n \in \{0, 1\}^{r \times r}$ = matrix of second components of $B^n$, i.e.,
  \[ (S_n)_{ij} = ((B^n)_{ij})_2 \quad (i, j \in [r]). \]
Splitting up a matrix over $\mathcal{D}$

Let $B \in \mathcal{D}^{r \times r}$.

- $F \in \mathbb{R}_{\text{max}}^{r \times r} =$ matrix of first components of $B$, i.e.,
  
  $$F_{ij} = (B_{ij})_1 \quad (i, j \in [r]).$$

- $S_n \in \{0, 1\}^{r \times r} =$ matrix of second components of $B^n$, i.e.,
  
  $$(S_n)_{ij} = ((B^n)_{ij})_2 \quad (i, j \in [r]).$$

Note:

- $F^n = (B^n)_1 \quad (n \in \mathbb{N})$ where addition and multiplication on the left hand side is performed in $\mathbb{R}_{\text{max}}$. 

Every cycle of maximum mean weight in $G(B)$ is a cycle of maximum mean weight in $G(F)$(some more technical effort is needed to define the mean weight of a cycle in $G(B)$).
Splitting up a matrix over $\mathcal{D}$

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Almost linear periodicity of the matrix of first components
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Lemma

Let $B \in \mathcal{D}^{r \times r}$ be irreducible.

- $F$ is almost linear periodic.

Moreover, $l_{\text{per}}(F)$ and $\lambda(B)$ can be computed in $O(r^3)$ time.

The sequence $(S_n)_{n \in \mathbb{N}}$ is ultimately constant, i.e., there is some $M \in \mathbb{N}$ with $S_n = S_M (n \geq M)$. 

$B$ is almost linear periodic, $\lambda(B)$ is the first component of a linear factor of $B$ and we have $l_{\text{def}}(F) \leq l_{\text{def}}(B) \leq \max\{M, l_{\text{def}}(F)\}$. 
Almost linear periodicity of the matrix of first components

Lemma

Let \( B \in D^{r \times r} \) be irreducible.

- \( F \) is almost linear periodic. We have

\[
\text{lfac}(F)_{ij} = \lambda(B)_1 > 0 \quad (i, j \in [r]).
\]

Moreover, \( \text{lper}(F) \) and \( \lambda(B) \) can be computed in \( O(r^3) \) time.
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\[ \text{ldef}(F) \leq \text{ldef}(B) \leq \max\{M, \text{ldef}(F)\}. \]
Our third primitivity criterion

Theorem
Let \( A \in \mathcal{P}[X]^{r \times r} \) have positive degree. Then the following statements are equivalent:

(i) \( A \) is \( \mathcal{P}_+[X] \)-primitive.

(ii)
\( \kappa_0(A) \) is \( \mathcal{P}_+ \)-primitive.
\( \kappa_1(A) \) is nonzero.
\( A \) is \( \mathcal{P}_+\mathcal{X} \)-irreducible.
\( (\lambda(s(A))^1,1) \) is a linear factor of \( s(A) \) and \( \lambda(s(A))^1 > 0 \).

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\( \kappa_0(A) \) is \( \mathcal{P}_+ \)-primitive.
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Miscellaneous remarks

- Let \( A \in P[X]^{r \times r} \) with \( \text{deg}(A) > 0 \). If \( A \) is \( P_+[X] \)-primitive then

\[
s(A^{\text{ldef}(s(A)) + \text{lper}(s(A))}) \in Q^{r \times r}.
\]

It is well-known that every irreducible matrix over the standard max-plus algebra has a unique eigenvalue (see Baccelli – Cohen – Olsder – Quadrat, Synchronization and Linearity (1992)).

On the other hand, the irreducible matrix

\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & -\infty \\
0 & 0
\end{pmatrix}
\]

\( \in D^{2 \times 2} \) does not have an eigenvalue.
Let $A \in \mathcal{P}[X]^{r \times r}$ with $\deg(A) > 0$. If $A$ is $\mathcal{P}_+[X]$-primitive then
\[
s(A^{\text{ldef}}(s(A)) + \text{lper}(s(A))) \in Q^{r \times r}.
\]
The converse does not hold: Let $A = X^2 + X \in \mathbb{R}_{\geq 0}[X]$. Then
\[
s(A) = (2, 1) \in Q, \quad \text{ldef}(s(A)) = 0, \quad \text{lper}(s(A)) = 1,
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Let $A \in \mathcal{P}[X]^{r \times r}$ with $\deg(A) > 0$. If $A$ is $\mathcal{P}_+[X]$-primitive then

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$$s(A) = (2, 1) \in \mathbb{Q}, \quad \text{ldef}(s(A)) = 0, \quad \text{lper}(s(A)) = 1,$$

but $A$ is not $\mathbb{R}_{> 0}[X]$-primitive.
Let $A \in \mathcal{P}[X]^{r \times r}$ with $\deg(A) > 0$. If $A$ is $\mathcal{P}_+[X]$-primitive then

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It is well-known that every irreducible matrix over the standard max-plus algebra has a unique eigenvalue (see Baccelli – Cohen – Olsder – Quadrat, Synchronization and Linearity (1992)).
Let $A \in \mathcal{P}[X]^{r \times r}$ with $\deg(A) > 0$. If $A$ is $\mathcal{P}_+[X]$-primitive then

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The converse does not hold: Let $A = X^2 + X \in \mathbb{R}_{\geq 0}[X]$. Then

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It is well-known that every irreducible matrix over the standard max-plus algebra has a unique eigenvalue (see Baccelli – Cohen – Olsder – Quadrat, Synchronization and Linearity (1992)). On the other hand, the irreducible matrix

$$\begin{pmatrix} (0, 0) & (1, 0) \\ (0, 0) & (-\infty, 0) \end{pmatrix} \in \mathcal{D}^{2 \times 2}$$

does not have an eigenvalue.
The maximal cycle mean weight need not be a linear factor of an irreducible matrix: The matrix

$$B = \begin{pmatrix} e & (1, 0) \\ \varepsilon & e \end{pmatrix} \in \mathcal{D}^{2 \times 2}$$

is irreducible, and we have $\lambda(B) = (\frac{1}{2}, 0)$,
The maximal cycle mean weight need not be a linear factor of an irreducible matrix: The matrix

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A conjecture on the primitivity exponent

**Conjecture**

If $A \in \mathcal{P}[X]^{r \times r}$ is $\mathcal{P}_+[X]$-primitive then we have

$$\gamma_{\mathcal{P}_+[X]}(A) \leq \max \{2 \deg (A) - 1, 4r^2(r - 1) + 1\}.$$