On the Number of Partitions of an Integer in the $m$-Bonacci Base

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Today, we consider representations of positive integers in the \(m\)-bonacci base.

We will see a formula for the number of representations in terms of binomial coefficients modulo 2.

Joint work with Luca Zamboni.
For each $m \geq 2$, we define the $m$-bonacci numbers by $F_k = 2^k$ for $0 \leq k \leq m - 1$ and $F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m}$ for $k \geq m$.

When $m = 2$, these are the usual Fibonacci numbers.
Generally, if we have $U = (u_n)_{n \geq 0}$, a strictly increasing sequence with $u_0 = 1$, then a representation of $n$ is a finite word $d_m d_{m-1} \cdots d_1 d_0$ so that

$$n = \sum_{k=0}^{m} d_k u_k.$$ 

The representation is written then as $d_m d_{m-1} \cdots d_1 d_0$, the most significant digit first.
Consider the Fibonacci sequence \((F_n)_{n \geq 0}\) beginning 1, 2, 3, 5, 8, 13, 21, 34, . . . .

Consider the integer 31.

We have that a representation of 31 is 1010010 and another is 1001110.
The Greedy Algorithm

- A greedy representation of a non-negative integer $n$ is a word $d_m \cdots d_0$, satisfying $d_m > 0$, $d_j \geq 0$ for $0 \leq j < m$, and for each $j$, $0 \leq j \leq m$, $d_j u_j + \cdots + d_0 u_0 < u_{j+1}$.

- This representation yields the largest representation with respect to the lexicographic order.

Example

The representation of 31 in the Fibonacci base, 1010010, is the representation of 31 obtained by the greedy algorithm.

$$31 = 1(21) + 0(13) + 1(8) + 0(5) + 0(3) + 1(2) + 0(1)$$
In the case of Fibonacci, the greedy representation is often called the Zeckendorff representation of $n$, based on a theorem in a paper published in 1972 by Edouard Zeckendorff.

This theorem proves that we can always get a unique representation of $n$ as a sum of distinct, non-consecutive Fibonacci numbers.

So we can get a representation of $n$ containing no occurrences of 11, and it is the Zeckendorff representation of $n$.

Generally, for $m$-bonacci, the greedy representation contains no occurrences of $1^m$. We call it the $m$-Zeckendorff representation of $n$, and write $Z_m(n)$.
Because of the recurrence relation \( F_k = F_{k-1} + F_{k-2} + \cdots + F_{k-m} \) for \( k \geq m \), given a representation of \( n \), we may obtain another representation via the exchange rule \( 10^m \equiv 01^m \).

When \( n = 50 \) and \( m = 2 \), there are the following 6 representations (arranged in decreasing lexicographic order):

- 10100100
- 10100011
- 10011100
- 10011011
- 1111100
- 1111011
Some Results

- In 1968, Carlitz derives formulas related to the number of distinct representations of an integer in the Fibonacci number system.
- In 2001, Jean Berstel gives a formula in terms of $2 \times 2$ matrices.
- In 2007, P. Kocábová, Z. Masáková, and E. Pelantová, extend Berstel’s result to $m$-bonacci numbers.
The Function that Counts the Number of Representations

- $R_m(n) =$ the number of representations of $n$ in the $m$-bonacci base.

- In the previous example, $R_2(50) = 6$.

- Observe that a number $n$ has a unique representation in the $m$-bonacci base if and only if $Z_m(n)$ does not contain any occurrences of $0^m$.

- In order to facilitate counting the number of representations, we consider a factorization of the greedy representation of $n$. 
The Principal Factorization of $Z_m(n)$

Either $Z_m(n)$ contains no occurrences of $0^m$ (in which case $R_m(n) = 1$), or $Z_m(n)$ can be factored uniquely in the form

$$Z_m(n) = V_1 U_1 V_2 U_2 \cdots V_N U_N W$$

where

- $V_1, V_2, \ldots, V_N$ and $W$ do not contain any occurrences of $0^m$.
- $0^{m-1}$ is not a suffix of $V_1, V_2, \ldots, V_N$.
- Each $U_i$ is of the form
  $$U_i = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m$$

  with $x_i \in \{0, 1\}$. 
Example for $m = 2$ (Fibonacci)

Let $w = 1000001000000100010010101010$. 

Then $w$ is a Zeckendorff representation for some integer $n$.

Factor $w$ as follows:

\[ w = (1000001000000)(10001000100)(101010) = U_1 U_2 W. \]

\[ R_2(w) = R(U_1) R(U_2). \]
We get new representations in the Tribonacci base by exchanging a 1000 for a 0111.

Let \( w = 101001000101010000010000010010 \).

We can factor \( w \) in the following way.

\[
w = (10)(1001000)(1010)(1000001000000)(10010) = V_1 U_1 V_2 U_2 W.
\]

Then \( R_3(w) = R(U_1)R(U_2) \).
Let $Z_m(n) = V_1 U_1 V_2 U_2 \cdots V_N U_N W$ be the principle factorization.

Then $R_m(n) = \prod_{i=1}^{N} R_m(U_i)$.

What remains is to compute $R_m(U_i)$ for each $i$. 
We now code $U = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m$ by a positive integer $r$ whose base 2 expansion is $1x_kx_{k-1} \cdots x_0$.

So $r = 1 \cdot 2^{k+1} + x_k \cdot 2^k + \cdots + x_1 \cdot 2 + x_0$.

Then $R_m(U) = \sum_{j=0}^r \binom{2r-j}{j}$ (mod 2).
An Example

- Let \( m = 2 \) so that we are in the case of Fibonacci and let \( n = 201 \). Then \( Z(n) = 10100000100 \).

- So \( r = 25 \) and has base 2 expansion 11001.

- Using the formula, \( R_2(n) = \sum_{j=0}^{25} \binom{50-j}{j} \pmod{2} = 12 \).
Consider a natural decomposition of the set of all partitions of \( n \) in the \( m \)-bonacci base.

Let \( F \) be the largest \( m \)-bonacci number less or equal to \( n \).

Set \( R_m^+(n) \) = the number of partitions involving \( F \) and \( R_m^-(n) \) = the number of partitions that do not.

Clearly \( R_m(n) = R_m^+(n) + R_m^-(n) \).
An Example

If $n = 50$ and $m = 2$, we have 6 representations:

- 10100100
- 10100011
- 10011100
- 10011011
- 1111100
- 1111011

- $R_2^+(50) = 4$ and $R_2^-(50) = 2$. 
Recursive Relations for $R_m(n)$

Let $U = 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m$ with $x_i \in \{0, 1\}$. Then

- $R_m^+(10^{m-1} 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m) = R_m^+(U) + R_m^-(U)$
- $R_m^-(10^{m-1} 10^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m) = R_m^-(U)$
- $R_m^+(10^{m-1} 00^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m) = R_m^+(U)$
- $R_m^-(10^{m-1} 00^{m-1} x_k 0^{m-1} x_{k-1} \cdots 0^{m-1} x_0 0^m) = R_m^+(U) + R_m^-(U)$
Define a Tower

- Let $U = 10^{m-1}x_k0^{m-1}x_{k-1} \cdots 0^{m-1}x_00^m$ with $x_i \in \{0, 1\}$.

- Construct a tower of $k + 2$ levels $L_0, L_1, \ldots, L_{k+1}$ where each $L_i$ consists of an ordered pair $(a,b)$ of positive integers.

- Start with $L_0 = (1, 1)$. Then obtain $L_{i+1}$ from $L_i$ according to the value of $x_i$.

- Suppose $L_i = (a, b)$. Then if $x_i = 0$ then $L_{i+1} = (a, a + b)$, and if $x_i = 1$ then $L_{i+1} = (a + b, b)$.

- Using the recursive relations on the previous slide, one can show that $L_{k+1} = (R_m^+(U), R_m^-(U))$. 

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Example

- Take $m = 2$ and $U = 10x_20x_10x_000 = 100000100$.
- Then $x_0 = 1$, $x_1 = 0$, and $x_2 = 0$. So we have the tower
  \[(2, 5) \quad (2, 3) \quad (2, 1) \quad (1, 1)\].
- So $R_2^+(U) = 2$, $R_2^-(U) = 5$ and $R_2(U) = 7$. 

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By the well-known Fine and Wilf Theorem, given a pair of relatively prime numbers \((p, q)\) there exists a \(\{0, 1\}\)-word \(w\) of length \(p + q - 2\) (unique up to isomorphism) having periods \(p\) and \(q\). And, if \(p\) and \(q\) are both greater than 1, then \(1 = \gcd(p, q)\) is not a period, i.e., the word contains both 0’s and 1’s.

Zamboni and Tijdeman in 2003 give an algorithm constructing a word of maximal length having given periods so that the suffixes of \(w\) of lengths \(p\) and \(q\) begin in different symbols.

We denote by \(FW(p, q)\) this unique Fine and Wilf word relative to \((p, q)\) with the property that its suffix of length \(p\) begins in 0 and its suffix of length \(q\) begins in 1.
We now apply this to the ordered pair \((p, q) = (R^+_m(U), R^-_m(U))\).

In the Zamboni-Tijdeman paper, we find that
\[ FW(R^+_m(U), R^-_m(U))01 \] is given explicitly by the following composition of morphisms:
\[
FW(R^+_m(U), R^-_m(U))01 = \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)
\]
where
\[
\tau_0(0) = 0 \quad \tau_0(1) = 01 \\
\tau_1(0) = 10 \quad \tau_1(1) = 1.
\]
Let $\alpha(r)$ be the number of occurrences of 1 in $FW(R_m^+(U), R_m^-(U))01$ and $\beta(r)$ the number of 0’s.

$$R_m(U) = R_m^+(U) + R_m^-(U)$$
$$= R_m^+(U) + R_m^-(U) - 2 + 2$$
$$= |FW(R_m^+(U), R_m^-(U))| + 2$$
$$= |FW(R_m^+(U), R_m^-(U))01|$$
$$= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|$$
$$= |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1 + |\tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_0$$
$$= \alpha(r) + \beta(r)$$
$$= |\tau_1 \circ \tau_{x_0} \circ \tau_{x_1} \circ \cdots \circ \tau_{x_k}(01)|_1$$
$$= \alpha(2r + 1).$$


