Road Coloring

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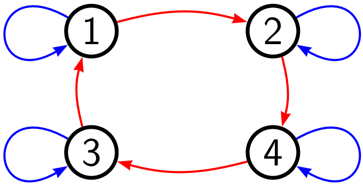
Roadmap of the talk

- Synchronizing words
- The Road Coloring Theorem
- The Road Coloring Theorem for periodic graphs
- Application to Huffman compression
- The hybrid Černý-Road Problem

Synchronizing words

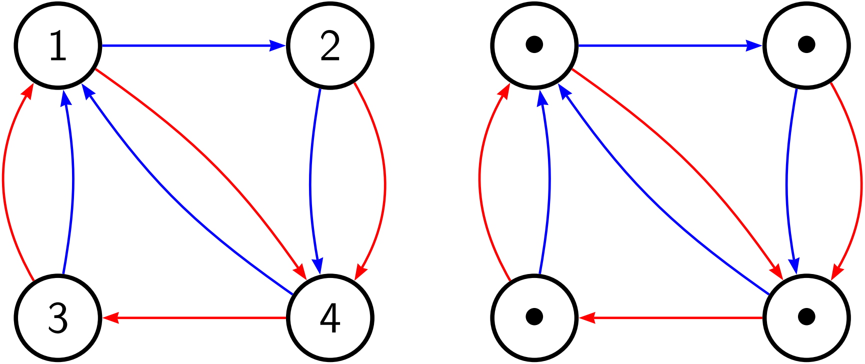


Synchronizing word (magic word, homing sequence, Rome word): a word w such that all paths labeled w terminate at the same vertex.

Synchronized automaton: automaton which has a synchronizing word.

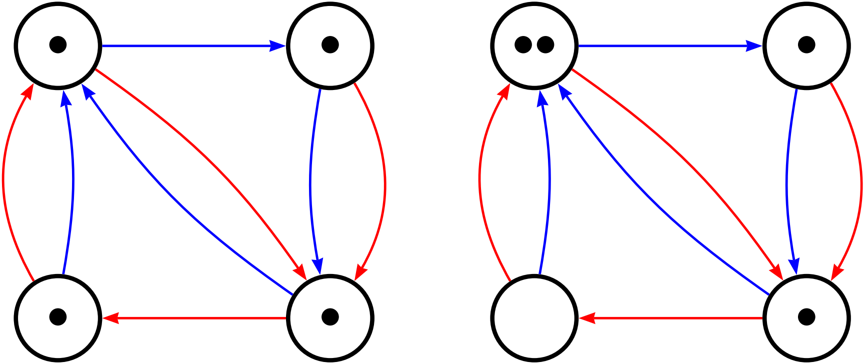
The automaton on the right is synchronized.
The word RRR is synchronizing.

The solitaire game



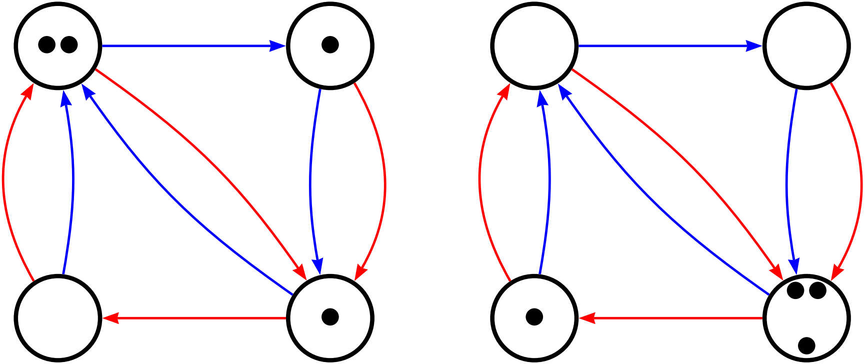
One pebble on each state.

The solitaire game: one plays B



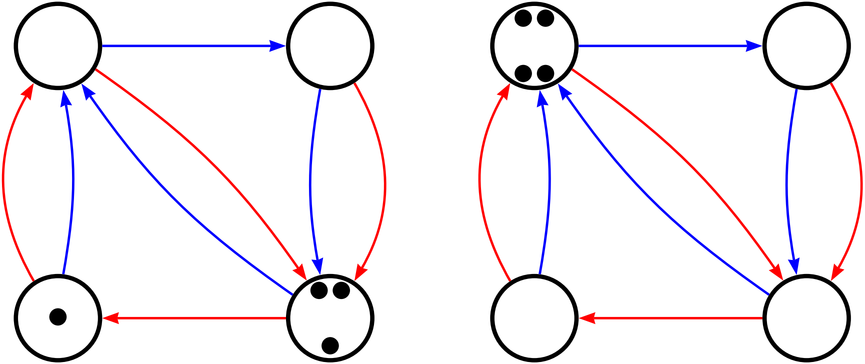
One plays B

The solitaire game: one plays R



One plays R

The solitaire game: one plays B



One plays B

The Road Coloring Problem (Adler,Goodwin, Weiss, 1977)

A directed graph with constant out-degree is road colorable if there is a coloring of its edges such that

- the edges going out of a vertex have distinct colors;
- there is a synchronizing word.

A graph is aperiodic if it is strongly connected and the gcd of the cycle lengths is 1.

A strongly connected graph which has a synchronized coloring is aperiodic.

A graph is admissible if it is strongly connected, aperiodic and has constant out-degree.

Theorem [A. Trahtman 2007]

Any admissible graph is road colorable.

Applications

- lossless source coding: the Huffman decoder can be chosen synchronized, hence resistant against errors
- communication protocols: test sequences to check whether a protocol conforms to its specification
- symbolic dynamics: for two aperiodic shifts of finite type X, Y with the same entropy, there exists a factor map $\varphi : X \to Y$ which is almost one-to-one. (invertible on "typical" sequences, *i.e.* on bi-infinite sequences which contain a synchronizing word infinitely often to the left and to the right).

Algorithm: the notion of stable pair

A synchronizable pair of states in an automaton is a pair of states (p, q) such that there is a word w with $p \cdot w = q \cdot w$.

A stable pair of states in an automaton is a pair of states (p, q) such that, for any word u, $(p \cdot u, q \cdot u)$ is a synchronizable pair.

The stable pair congruence is the relation defined on the set of states by $p \equiv q$ if (p, q) is a stable pair.

Lemma [Culik, Karhumäki, Kari, 2002]

If the quotient of $\mathcal{A}$ by a stable pair congruence is colorable, then $\mathcal{A}$ is colorable.

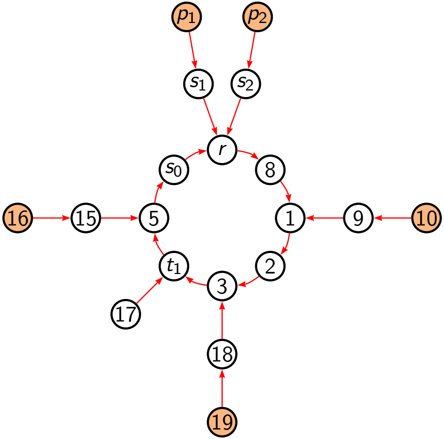
Algorithm for Road Coloring

FindColoring (aperiodic automaton $\mathcal{A}$, quotient automaton $\mathcal{B}$)

$\mathcal{B} \leftarrow \mathcal{A}$
while (size($\mathcal{B}$) > 1)
  do Update($\mathcal{B}$)
    $\mathcal{B}, (s, t) \leftarrow$ FindStablePair($\mathcal{B}$)
    lift the coloring up from $\mathcal{B}$ to the automaton $\mathcal{A}$
    $\mathcal{B} \leftarrow$ Merge($\mathcal{B}, (s, t)$)
return $\mathcal{A}$

Finding a coloring which has a stable pair

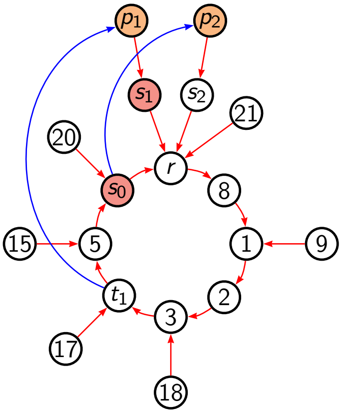
We start with some coloring and fix a color (red).
The level of a state is its distance to the red cycle of its cluster.
Maximal states are states of maximal level.



Finding a coloring which has a stable pair

Lemma [Trahtman 2007]

If all maximal states belong to the same tree, then there is a stable pair.



Proof of Trahtman's lemma

Lemma [Trahtman 2007]

If all maximal states belong to the same tree, then there is a stable pair.

A minimal image $I = Q \cdot w$ is an image minimal for set inclusion. For any word u, we have $I \cdot u$ is a minimal image.

Proof.

Let I be a minimal image and ℓ be the maximal level.
By irreducibility, it contains a maximal state p in a tree on a cycle C.
If there is $q \neq p$ maximal in I, then $|I \cdot a^{\ell}| < |I|$ (contradiction).
Let m be a common multiple of the lengths of all red cycles.
Let s_0 be the predecessor of r in C and s_1 the child of r containing p.
Let $J = I \cdot a^{\ell-1}$ and $K = J \cdot a^m$.
We have $J = \{s_1\} \cup R$ with $R \subset$ 0-level and $K = \{s_0\} \cup R$.

□

Proof of Trahtman's lemma (2)

Lemma [Trahtman 2007]

If all maximal states belong to the same tree, then there is a stable pair.

A minimal image $I = Q \cdot w$ is an image minimal for set inclusion. For any word u, we have $I \cdot u$ is a minimal image.

Proof.

Let s_0 be the predecessor of r in C and s_1 the child of r containing p.
Let $J = I \cdot a^{\ell-1}$ and $K = J \cdot a^m$.
We have $J = \{s_1\} \cup R$ with $R \subset$ 0-level and $K = \{s_0\} \cup R$.
Let w a word of minimal rank. For any word v, $|J \cdot vw| = |K \cdot vw| = |I|$.
We claim that the set $(J \cup K) \cdot vw$ is a minimal image.
Indeed, $J \cdot vw \subseteq (J \cup K) \cdot vw \subseteq Q \cdot vw$. (all 3 are equal).
But $(J \cup K) \cdot vw = R \cdot vw \cup s_0 \cdot vw \cup s_1 \cdot vw$.
This forces $s_0 \cdot vw = s_1 \cdot vw$.
Thus (s_0, s_1) is a stable pair.

□

Finding a stable pair with a sequence of flips

A flip: an exchange of the labels (with one a) of two edges going out of some state.

Make a sequence of flips such that

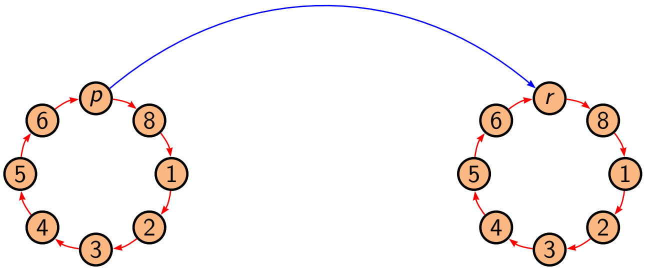
- either all maximal states belong to a same tree,
- or the number N_0 of 0-level states increases

We consider several cases corresponding to the geometry of the automaton.

Finding a stable pair with a sequence of flips (2)

Case 1 : The maximal level is zero

- If the set of outgoing edges of each state is a bunch, then there is only one red cycle, the graph is not aperiodic.
- Let p with $p \xrightarrow{a} q$ and $p \xrightarrow{b} r$ and $q \neq r$. We flip these edges. We get a unique maximal tree, hence a stable pair.



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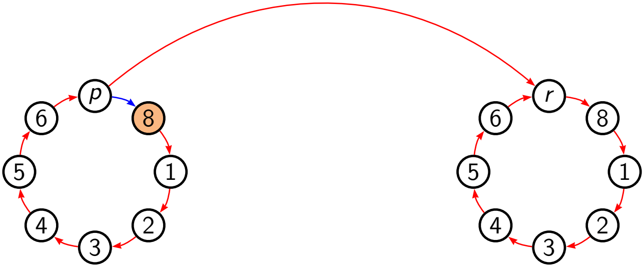


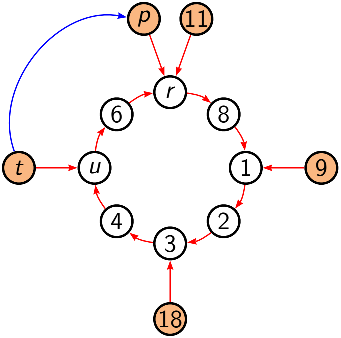
Finding a stable pair with a sequence of flips (3)

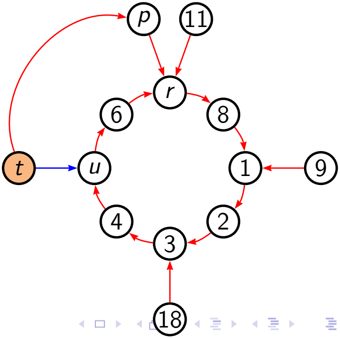
Case 2 : The maximal level is $\ell > 0$.

Let p maximal, r its root, and $t \xrightarrow{b} p$.

We denote $u = t \cdot a$.

- Case 2.1. If t is not in the same cluster as r, or if t has a positive level and does not belong to the a-path from p to r, we flip $t \xrightarrow{b} p$ and $t \xrightarrow{a} u$ and get an automaton which has a unique maximal tree.





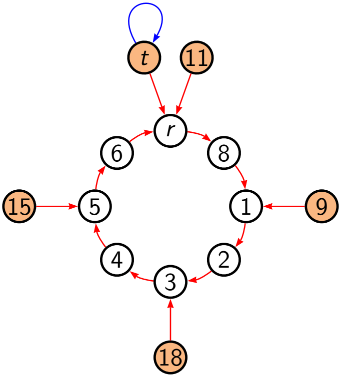
Finding a stable pair with a sequence of flips (4)

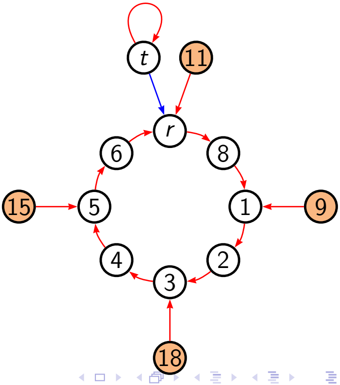
Case 2 : The maximal level is $\ell > 0$.

Let p maximal, r its root, and $t \xrightarrow{b} p$.

We denote $u = t \cdot a$.

- Case 2.2. If t belongs to the a-path from p to r, we flip $t \xrightarrow{b} p$ and $t \xrightarrow{a} u$ and increase the number of N_0 of 0-level states.

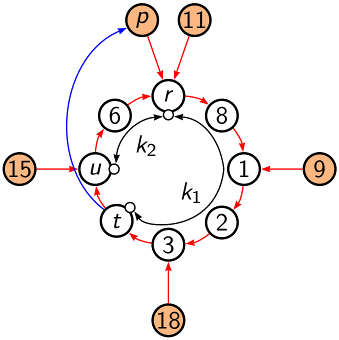


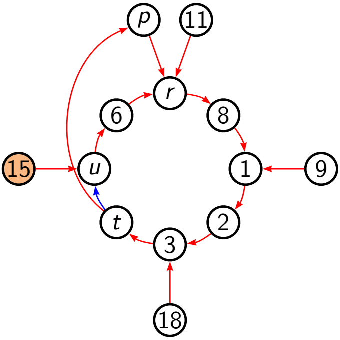


Finding a stable pair with a sequence of flips (5)

Case 2 : The maximal level is $\ell > 0$.

- Case 2.3. We assume that t belongs to the cycle containing r. Let k_1 be the length of the simple a-path from r to t and k_2 the length of the simple a-path from u to r.
  - If $k_2 > \ell$, we flip the edges $t \xrightarrow{b} p$ and $t \xrightarrow{a} u$ and get an automaton which has a unique maximal tree.

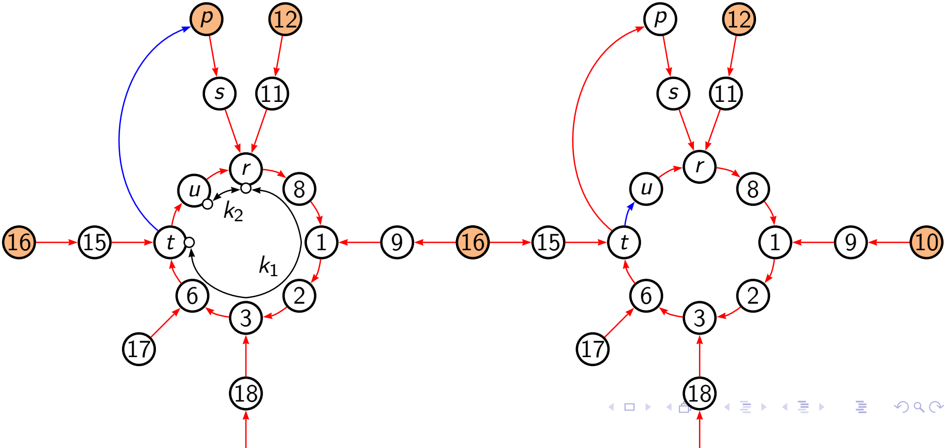




Finding a stable pair with a sequence of flips (6)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

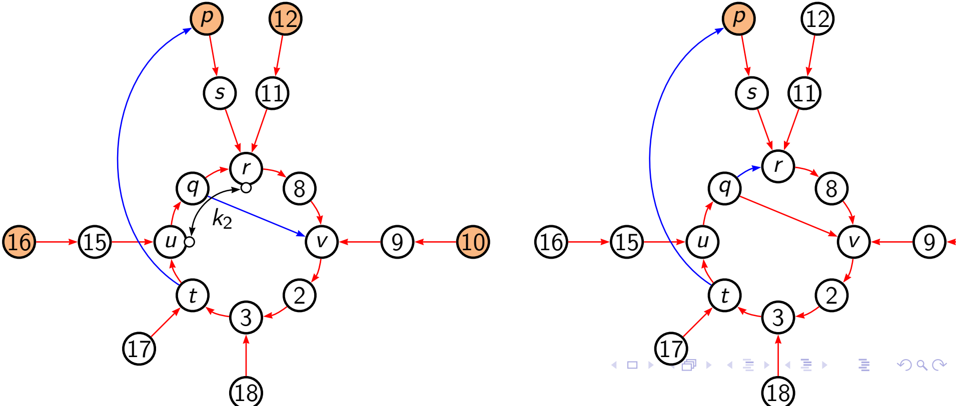
- If $k_2 < \ell$, we flip the edges $t \xrightarrow{b} p$ and $t \xrightarrow{a} u$ and get an automaton which has strictly more states of null level since $k_1 + \ell + 1 > k_1 + k_2 + 1$.



Finding a stable pair with a sequence of flips (7)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

- If $k_2 = \ell$, let q be the predecessor of r on the cycle, let s be the child of r ascendant of p in the maximal tree T.
    - If q has no bunch, there are edges $q \xrightarrow{a} r$ and $q \xrightarrow{c} v$ with $v \neq r$. We flip these edges. If r belongs to the new red cycle, N_0 increases. If not, level$(r) \geq 1$ in the new automaton and thus the new automaton has a unique maximal tree.



Finding a stable pair with a sequence of flips (8)

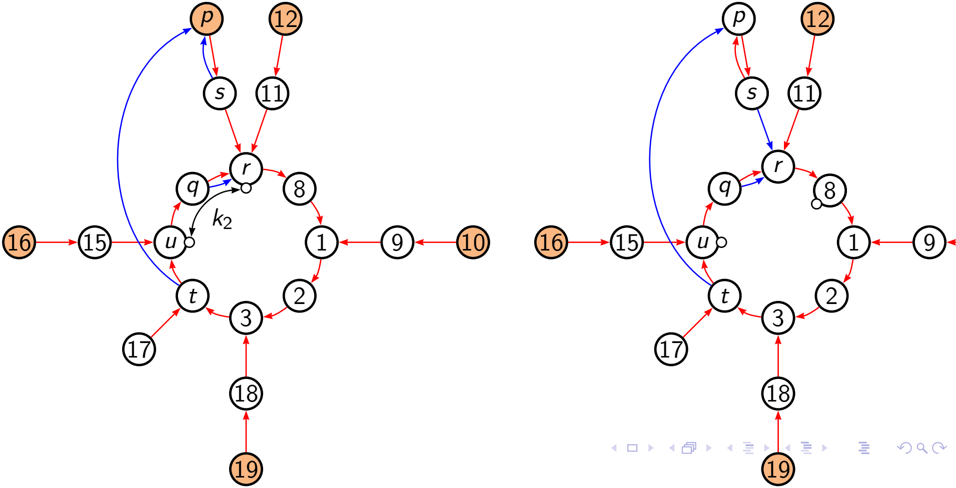
Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

- If $k_2 = \ell$, let q be the predecessor of r on the cycle, let s be the child of r ascendant of p in the maximal tree T.
  - If q and s have bunches, (q, s) is a stable pair.

Finding a stable pair with a sequence of flips (9)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

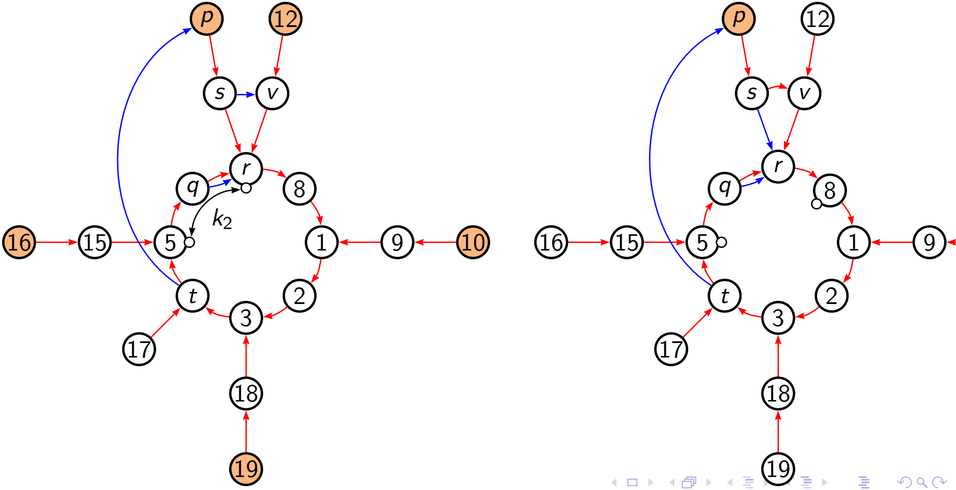
- We have $k_2 = \ell$. If q has a bunch and s not, there are edges $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$, with $v \neq r$. If there is an a-path from v to s, we flip the two edges, creating a new red cycle, which increases N_0.



Finding a stable pair with a sequence of flips (10)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

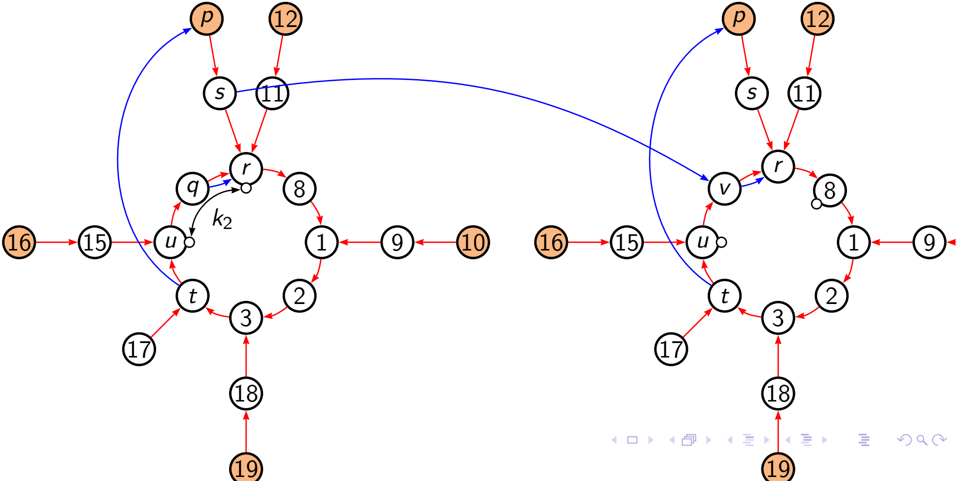
- We have $k_2 = \ell$. If q has a bunch and s not. There is no a-path from v to s and level(v) > 0, we flip $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$.



Finding a stable pair with a sequence of flips (11)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

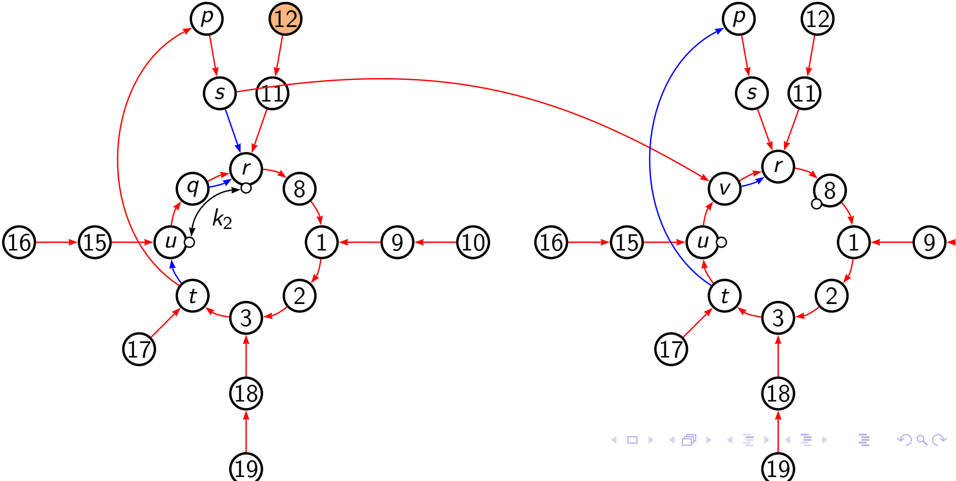
- We have $k_2 = \ell$. If q has a bunch and s not. We assume that level(v) $= 0$, and v is in another cluster, we flip $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$ and also $t \xrightarrow{a} u$ and $t \xrightarrow{b} p$.



Finding a stable pair with a sequence of flips (11)

Case 2.3 : The maximal level is $\ell > 0$, $t \in C$.

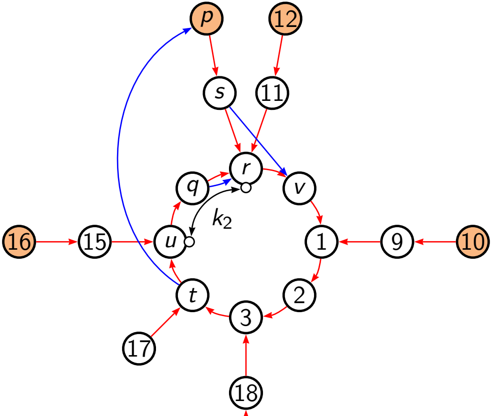
- We have $k_2 = \ell$. If q has a bunch and s not. We assume that level(v) $= 0$, and v is in another cluster, we flip $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$ and also $t \xrightarrow{a} u$ and $t \xrightarrow{b} p$.

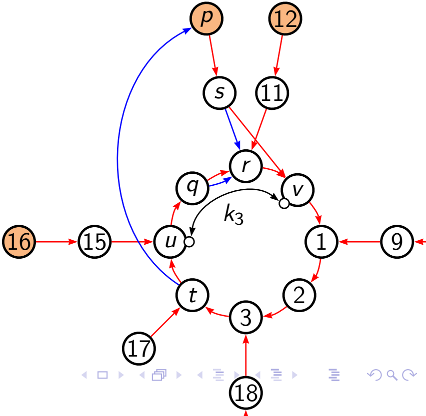


Finding a stable pair with a sequence of flips (12)

Case 2.3.3 : The maximal level is $\ell > 0$, $t \in C$, $k_2 = \ell$

- If q has a bunch and s not, and there are edges $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$, with $v \neq r$ in C. Let us denote by k_3 the length of the simple a-path from u to v.
  Since $v \neq r$, we have $k_3 \neq k_2$. Hence $k_3 < \ell$ or $k_3 > \ell$. We flip the edges $s \xrightarrow{a} r$ and $s \xrightarrow{c} v$ and proceed as in Case 2.3.1 or 2.3.2.





Complexity of the Road Coloring problem

The complexity of Trathman's algorithm is cubic $O(\text{card}\, A \times n^3)$.

Theorem [B., Perrin 2008 preprint]

One can compute a synchronized coloring of an n-state admissible graph in time $O(\text{card}\, A \times n^2)$.

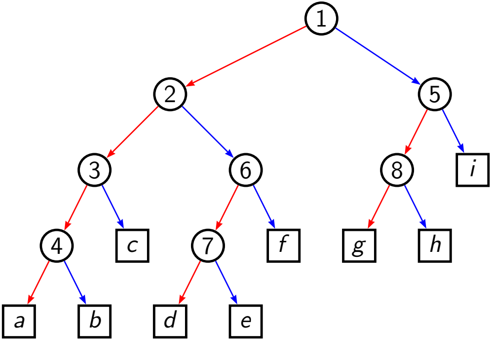
The Road Coloring Problem for periodic graphs

The period of a graph is the gcd of the lengths of the cycles.
The rank of a colored graph (Q, E) is the minimal cardinality of the sets $Q \cdot u$ for all colored sequences u.

Theorem [B., Perrin preprint 2008, Budzban, Feinsilver 2011]

Any strongly connected with constant outgoing arity has a coloring whose rank is equal to the period of the graph.

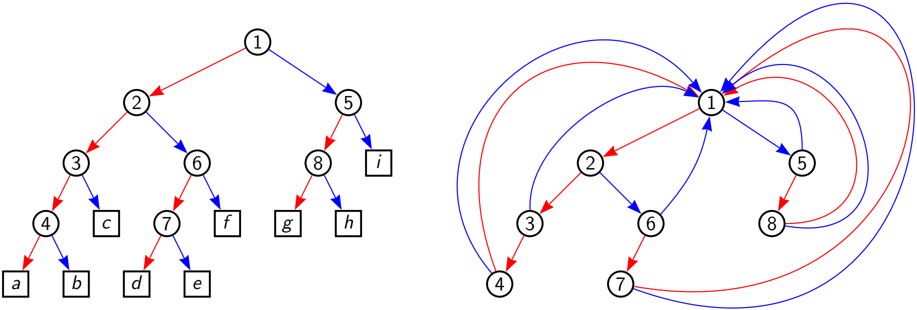
Application to Huffman compression



a : 1/16, b : 1/16, c : 1/8, d : 1/16, e : 1/16, f : 1/8, g : 1/8, h : 1/8, i : 1/4

Encoding: $d \leftrightarrow$ RBRR

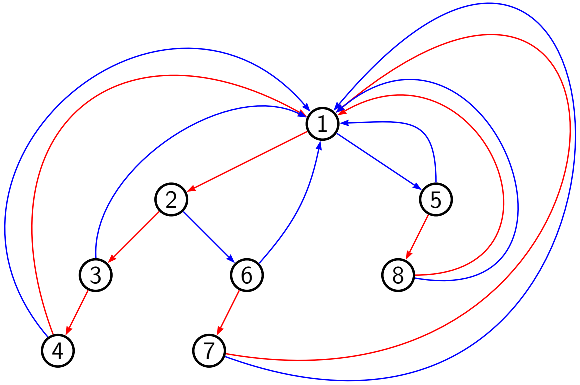
Application to Huffman compression (2)



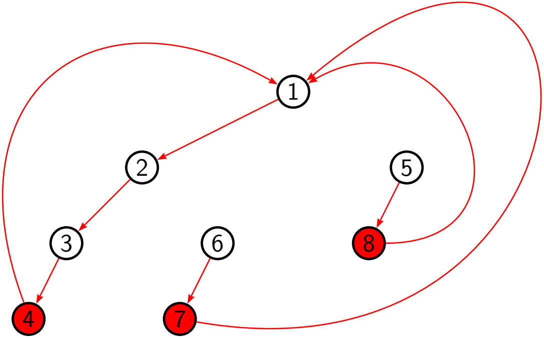
A non synchronized Huffman decoder.

When the lengths of the codewords in are relatively prime, there is another Huffman code (with the same length distribution) with a synchronized decoder.

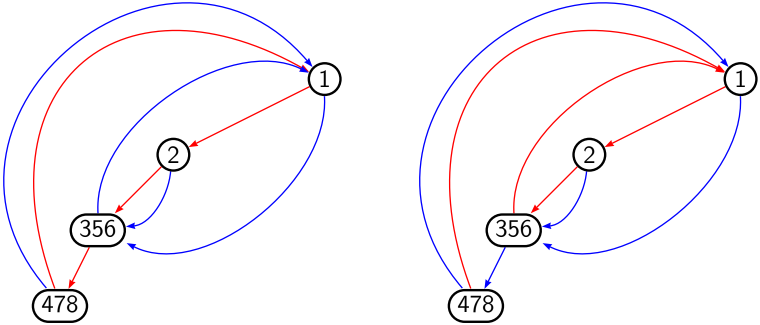
Application to Huffman compression (3)



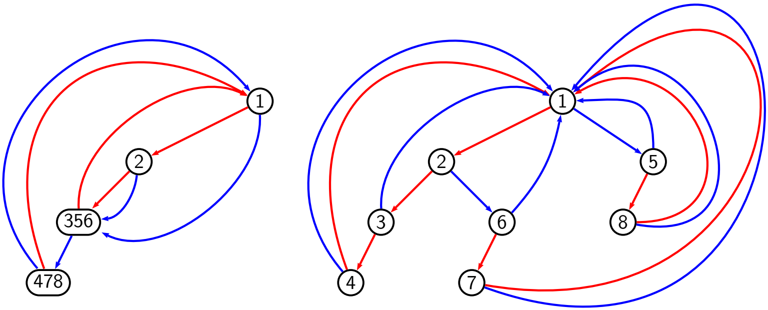
Application to Huffman compression (3)



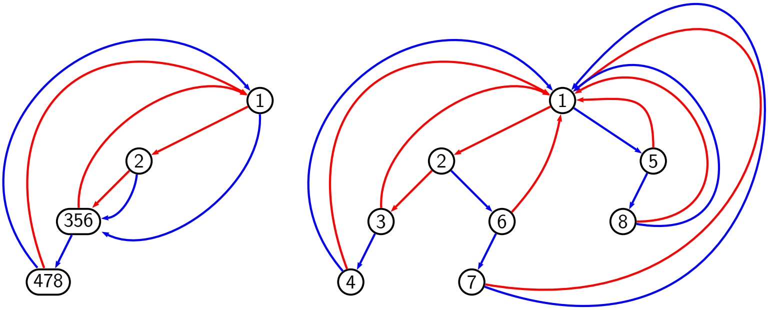
Quotient + flip at 356



Lifting up the flips



Lifting up the flips



$a \leftrightarrow$ RRBR

$b \leftrightarrow$ RRBB

$c \leftrightarrow$ RRR

$d \leftrightarrow$ RBR

...

The sequence RBR is a homing sequence.

The Road Coloring algorithm makes the Huffman decoder synchronized.

The hybrid Černý-Road Problem

Volkov raised the following question:

What is the minimum length of a synchronizing word for a synchronized coloring of an aperiodic graph?

We conjecture that a synchronized coloring such that the coloring is moreover one-cluster can be obtained with the same complexity. This guaranties a minimum length of a synchronizing word of a length at most quadratic in the number of vertices.