Substitutions, Rauzy fractals and Tilings

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CANT, 2009
Several questions

- **Geometry** Efficient way to produce a pixelized line? A pixelized plane?
- **Number theory** Compute good rational approximations of a vector?
- **Physics** Appropriate location for atoms to enhance conductivity properties and build frying pans?
- **Mathematics** Provide codings for toral translations?
Common (partial) answer

Behind all these questions, a similar problem:

Do fractal shapes generate a tiling of a plane?

Purpose of the lectures: exhibit several types of graphs and algorithms to stress out fractal properties...
Section 1: Basic definitions
Substitution / periodic point

- **Alphabet** \( \mathcal{A} = \{1, \ldots, n\} \).
- **Substitution**: replacement rule on \( \mathcal{A} \)
  \[
  \sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1
  \]
- **Periodic point**
  - Iterate the substitution until the image of a letter starts by the same letter.
    \[
    \sigma^d(a) = a,\ldots
    \]
  - Successively iterate \( \sigma^d \)

  
  1
  12
  12 13
  12 13 12 1
  12 13 12 1 12 13 12
  12 13 12 1 12 13 12 12 13 12 1 12 13

  - The limit is a **periodic point of the substitution**
    \[
    u = \sigma^\infty(a) \quad \sigma^k(u) = u
    \]
Primitivity

Definition
A substitution $\sigma$ is **primitive** if there exists $k$ such that all letters appears in the image of all letters through $\sigma^k$.

- **Examples**
  - $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$ is primitive.
  - $\sigma(1) = 12$, $\sigma(2) = 32$, $\sigma(3) = 23$ is not primitive.

- **Checking the property ?**
  - Incidence matrix $M_{\sigma}$ abelianization of the substitution
  - $\sigma$ primitive iff $\exists k$, $M_{\sigma}^k$ has only positive coefficients.

\[
M = \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
\end{pmatrix}
\]

$M^3 > 0$

\[
M = \begin{pmatrix}
  1 & 1 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 1 \\
\end{pmatrix}
\]

$M^n$ always has zeros in the first line.

- **Main interest** The matrix $M$ has a simple dominant eigenvalue
Primitivity

Theorem
Every substitution on a $n$-letter alphabet has at least one (and at most $n$) periodic points.

Proof:
- if $w$ is a periodic point, then $\sigma^k(u_0)$ starts with $u_0$.
- This means that there exists a loop starting from $u_0$ in the graph:
  \[ a \rightarrow b \text{ iff } \sigma(a) \text{ starts with } b \]
- The graph is finite and every node have an exiting edge.
  It contains at least one loop!
Answer Replace letters by as many independent vectors as needed in $\mathbb{R}^n$

- $e_1, \ldots, e_n$ canonical base of $\mathbb{R}^n$.
- Abelianization map $P : w_1 \ldots w_k \in A^n \mapsto e_{w_1} + \cdots + e_{w_k} \in \mathbb{R}^n$.

\begin{align*}
1211212112 \mapsto \quad & \quad \\
\end{align*}

**Commutation formula**: applying a substitution to a word is equivalent to applying the incidence matrix to its linearized vector.

\[ P(\sigma(W)) = MP(W) \]
Represent an infinite word?

**Question** How to see the fixed point of a substitution on a $n$-letters alphabet?

**Fixed infinite word $\rightarrow$ Stair in $\mathbb{R}^n$**

12 13 12 1 12 13 12 12 13 12 1 12 13

**Specific case:** the stair turns around a line and remains at a finite distance from it?
Definition
A substitution is unit of Pisot type if the dominant eigenvalue of its incidence matrix is a unit Pisot number (the determinant is $\pm 1$ and all the roots of the characteristic polynomial have a modulus less than or equal to 1).

In other words The matrix $\mathbf{M}_\sigma$ has one expanding eigenvalue $\beta$ and all the other eigenvalues $\beta^{(i)}$ are contracting.
Section 2: Rauzy fractal and main properties
Rauzy fractals

σ substitution unit of Pisot type on a \( n \) letters alphabet.

- \( \mathbb{H}_e \) expanding line of \( M_\sigma \).
- \( \mathbb{H}_c \) contracting hyperplane of \( M_\sigma \).
- \( \pi : \mathbb{R}^n \to \mathbb{H}_c \) projection along \( \mathbb{H}_e \).

Commutation relation: \( \pi M_\sigma \) (and its equivalent \( \pi P_\sigma \)) is a contraction \( h \) on the hyperplane \( \mathbb{H}_c \).
Rauzy fractals

Proposition

If $\sigma$ is unit of Pisot type, then the stair of any periodic point turn around the expanding line and remains at a finite distance from it.

Proof: A node the stair have the shape $\pi P(u_0 \ldots u_K)$.

- $u_0 \ldots u_k \ldots$ is fixed by $\sigma^d$.

$$\sigma^d(u_0 \ldots u_l) \leq_{\text{lex}} u_0 \ldots u_l <_{\text{lex}} \sigma^d(u_0 \ldots u_{l+1})$$

- There exists a prefix $p$ of image of a letter under $\sigma^d$ such that

$$u_0 \ldots u_k = \sigma^d(u_0 \ldots u_l)p$$

- Iterate this process

$$u_0 \ldots u_k = \sigma^{dN}(p_N)\sigma^{d(N-1)}(p_{N_1}) \ldots \sigma^d(p_1)p_0$$

- Abelianize this relation

$$\pi P(u_0 \ldots u_k) = h^{dN}P(p_N) + h^{d(N-1)}P(p_{N_1}) \ldots + h\sigma^dP(p_1) + P(p_0)$$

- $h$ is a contraction and the $p_k$'s are in finite number

The projections of the stair along the expanding line are bounded
Rauzy fractals

Definition (Rauzy fractal / central tile)
Project the nodes of the stair and take the closure.

\[ T := \{ \pi(P(u_0 \ldots u_{N-1})) \mid N \in \mathbb{N} \} \]

Subtiles
Look at the last edge to be read.

\[ T(i) := \{ \pi(P(u_0 \ldots u_{k-1})) \mid N \in \mathbb{N}, \ u_N = i \} \]
Basic properties

Theorem

A Rauzy fractal is compact and it is independant from the periodic point that was chosen.

Proof

- All substitutions of Pisot type are primitive.
- The primitivity assumption implies that all periodic points have the same set of finite factors.
- The closure operation in the definition implies that the Rauzy fractal only depends on factors of the periodic point.
Basic properties

Theorem
The measures of the subtiles are proportional to the expanding eigenvector of the incidence matrix.

Proof
- Desubstitution $u_1 \ldots u_k \ is = \sigma(u_1 \ldots u_l) \sigma(j) = p i s$
- Set decomposition $\mathcal{T}(i) = \bigcup_{\sigma(j) = \text{pis}} \mathcal{h} \mathcal{T}(j) + \pi P(p)$.
- $h$ is a contraction whose ratio is the product of conjugate of the dominant eigenvalue $\beta$.

$$\mu(\mathcal{T}(i)) \leq 1/\beta \sum_{\sigma(j) = \text{pis}} \mu(\mathcal{T}(j)) \quad (\mu(\mathcal{T}(i)))_i \leq 1/\beta \mathbf{M}(\mu(\mathcal{T}(i)))_i$$

- Perron-Frobenius theorem. Assume that $\mathbf{M}$ is a matrix with positive entries. Let $\beta$ be its dominant eigenvalue. Let $\mathbf{X}$ be a vector with positive entries. Then we have $\mathbf{MX} \leq \beta \mathbf{X}$. The equality holds only when $\mathbf{X}$ is a dominant eigenvector of $\mathbf{M}$. 
Decomposition

\[ T(i) = \bigcup_{\sigma(j)=pis} hT(j) + \pi P(p). \]
\[ \sigma(1) = 12, \ \sigma(2) = 13, \ \sigma(3) = 1. \]

\( h \) is a contracting similarity of angle \( \simeq 2/3 \)

- The letter 1 appears in the images of 1, 2 and 3, with no prefix behind.
  \[ T(1) = hT(1) \cup hT(2) \cup hT(3) \]

- The letter 2 appear only in the image of 1, behind a prefix 1.
  \[ T(2) = hT(1) + \pi P(1) \]

- The letter 3 appear only in the image of 2, behind a prefix 1.
  \[ T(3) = hT(2) + \pi P(1) \]
GIFS: telling which tile to put in the tile

\[ T(i) = \bigcup_{\sigma(j)=pis} hT(j) + \pi P(p). \]

- **Graph directed iterated function system.** \((G, \{\tau_e\}_{e \in E})\)
  - Finite directed graph \(G\) with no stranding vertices
  - With each edge \(e\) of the graph, is associated a contractive mapping \(\tau_e : \mathbb{R}^n \rightarrow \mathbb{R}^n\).

- **GIFS attractors** unique compact sets \(K_i\) such that
  \[ K_i = \bigcup_{i \rightarrow j} \tau_e(K_j) \]

- **Prefix-suffix graph** : defined naturally from the decomposition formula

\[ i \xrightarrow{(p,i,s)} j \text{ iff } \sigma(j) = pis \]
Theorem (Arnoux&Ito’01, Sirvent&Wang’02)

Let $\sigma$ be a primitive unit Pisot substitution over the alphabet $A$ of $n$ letters. The subtiles of $T$ are solutions of the GIFS, with measure disjoint sets:

$$\forall i \in A, \quad T(i) = \bigcup_{j \in A, (p,i,s) \rightarrow j} hT(j) + \pi P(p).$$

The central tile $T$ is a compact subset of $\mathbb{R}^{n-1}$ with non-empty interior. Each subtile is the closure of its interior.

Self-similar? $h$ must have eigenvalues with the same modulus.
Condition for disjointness?

From the GIFS, the pieces are disjoint in the decomposition of each subtile

\[ \mathcal{T}(i) = \bigcup_{j \in A, (p, i, s)} j \mathcal{T}(j) + \pi \mathcal{P}(p). \]

Can we deduce that the subtiles are also disjoint?

Yes... when the subtiles are properly located in the decomposition!
Condition for disjointness

Definition
A substitution $\sigma$ over the alphabet $A$ satisfies the strong coincidence condition if for every $(j_1, j_2) \in A^2$, we have $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $P(p_1) = P(p_2)$ or $P(s_1) = P(s_2)$.

Theorem (Arnoux&Ito’01)
If $\sigma$ satisfies the strong coincidence condition, then the subtiles $T(i)$ of the central tile $T$ have disjoint interiors.

Proof: The pieces $hT(j_1) + P(p_1)$ and $hT(j_2) + P(p_2)$ both appear in the GIFS decomposition of $T(i)$.

Effectivity? Every substitution over a 2-letter alphabet has strong coincidence (Diamond&Barge’02). No counter-example is known in general.
Section 3: Tilings
Definitions

**Definition (multiple tilings)**

Let $K_i$, $i \in A$ be a finite collection of compact sets of a Euclidean space $\mathbb{H}$.

A *multiple tiling* of the space $\mathbb{H}$ by the compact sets $K_i$ is given by a translation set $\Gamma \subset \mathbb{H} \times A$ such that

- $\mathbb{H} = \bigcup_{(\gamma, i) \in \Gamma} K_i + \gamma$,
- each compact subset of $\mathbb{H}$ intersects a finite number of tiles,
- almost all points in $\mathbb{H}$ are covered exactly $p$ times for some positive integer $p$.

If $p = 1$, then the multiple tiling is called a *tiling*. 
Tiling of the expanding line

- Let $\sigma$ be a substitution unit of Pisot type and let $u$ be a periodic point.
- Project the stair of $u$ on the expanding line along the contracting plane.

**Tiling of the expanding line!**

- Unique projection $\pi_e$ from $\mathbb{R}^n$ to $\mathbb{R}$ such that $\pi_e(H_c) = 0$ and $\pi_e(e_1) = 1$.

**Definition**

The **self-similar expanding tiling** for $\sigma$ is the set of interval in $\mathbb{R}$ whose endpoints belong to the projections under $\pi_e$ of the stair of the periodic point $u$. 
Why self-similar and expanding?

Self-similar with expansion factor $\beta$?

- The set of endpoints in stable by a multiplication by $\beta$
- The tiles are solutions of a GIFS where all the contractions are similarityed with the same ratio.

Here: endpoints are given by \( \{ \pi_e P(u_0 \ldots u_{k-1}) \mid k \geq 0 \} \)

- Commutation relation for $\pi_e$
- Decomposition of the endpoints by using prefixes (same as before)
Tiling based on the Rauzy fractal?

We have a Rauzy fractal and a decomposition rule.

Can we produce a tiling from these pieces?
How to define location of tiles?

- Arithmetic definition of the contacting plane $\mathbb{H}_c$?
  $\mathbb{H}_c$ is orthogonal to the expanding eigenvector $v_\beta$ of $t^* M_\sigma$

- Arithmetic discrete plane (Reveilles’91) : points with integer coordinates that approximate $\mathbb{H}_c$
  $$\{ x \in \mathbb{Z}^n | 0 \leq \langle x, v_\beta \rangle < \sum_{i \in A} \langle e_i, v_\beta \rangle \}.$$

- Face of type $i$ located in $x$ orthogonal to the $i$-th axis of a translate of the unit cube located at $x$.

Stepped hyperplane Union of faces of unit cubes whose set of vertices is the arithmetic discrete plane (Arnoux&Berthe&Ito’02)
$x$ is above $\mathbb{H}_c$ and $x - e_i$ is below $\mathbb{H}_c$. 

\[ \text{Stepped hyperplane Union of faces of unit cubes whose set of vertices is the arithmetic discrete plane (Arnoux&Berthe&Ito’02)} \]
\[ x \text{ is above } \mathbb{H}_c \text{ and } x - e_i \text{ is below } \mathbb{H}_c. \]
How to define location of tiles?

**Translation set** $\Gamma \subset \left\{ \begin{array}{c} \mathbb{H} \times A \\ \text{location of a tile} \\ \text{name of the tile to be drawn} \end{array} \right\}$

Stepped surface: Faces $[x, i^*]$ such that $x$ is above $\mathbb{H}_c$ and $x - e_i$ is below $\mathbb{H}_c$.

**Definition**
The *Self-replicating translation set* is the projection of the stepped surface on the contracting plane.

$$\Gamma_{sr} = \{[\gamma, i^*] \in \pi(\mathbb{Z}^n) \times A \mid \gamma = \pi(x), x \in \mathbb{Z}^n, 0 \leq \langle x, v_\beta \rangle < \langle e_i, v_\beta \rangle \}.$$
Why is it self-replicating?

Find a 2-dimensional substitution that stabilize $\Gamma_{sr}$?

Definition

The **GIFS 2D map** $\tilde{E}_1$ **of a substitution** $\sigma$ is defined on the subsets of $\pi(\mathbb{Z}^n) \times A$ as follows:

$$\tilde{E}_1\{[\gamma, i^*]\} = \bigcup_{j \in A, \sigma(j)=pis} \{[h^{-1}(\gamma + \pi\mathbf{P}(p)), j^*] \},$$

Mimic the GIFS rule in the Rauzy fractal!

$$h^{-1}T(1) = T(1) \cup T(2) \cup T(3)$$  
$$[0, 1^*] \mapsto [0, 1^*] \cup [0, 2^*] \cup [0, 3^*]$$  
$$h^{-1}T(2) = T(1) + \pi(e_3)$$  
$$[0, 2^*] \mapsto [\pi(e_3), 1^*]$$  
$$h^{-1}T(3) = T(2) + \pi(e_3)$$  
$$[0, 3^*] \mapsto [\pi(e_3), 2^*]$$
Why is it self-replicating?

**Theorem (Arnoux&Ito’01)**

Let $\sigma$ be a unit substitution of Pisot type. The images of two faces in $\Gamma_{sr}$ share no face in common. The translation set is stable under the action of the map $\tilde{E}_1$, which maps $\Gamma_{sr}$ onto $\Gamma_{sr}$.

$$\tilde{E}_1(\Gamma_{sr}) = \Gamma_{sr}$$

In other words. The projection of the stepped surface in a 2-dimensional fixed point for the GIFS map $\tilde{E}_1$.

The GIFS map $\tilde{E}_1$ plays the role of the multiplication by $\beta$ on the expanding tiling.

“Self-replicating set” since the map $h$ is not always a similarity.
Proposition

Let $\sigma$ be a primitive unit Pisot substitution. The self-replicating translation set $\Gamma_{sr}$ is repetitive and satisfies

$$\mathbb{H}_c = \bigcup_{[\gamma,i^*] \in \Gamma_{sr}} (\mathcal{T}(i) + \gamma)$$

where almost all points of $\mathbb{H}_c$ are covered $p$ times ($p$ is a positive integer).

In other words: we have defined a Self-replicating multiple tiling
Proof (step 1) : locally finite

\[ \mathcal{I} := \{ \mathcal{T}(i) + \gamma \mid [\gamma, i^*] \in \Gamma_{sr} \}. \]

As \( \mathcal{T}(i) \ (i \in A) \) are compact sets, i.e., there exists a \( P \in \mathbb{N} \) such that each point of \( \mathbb{H}_c \) is covered at most \( P \) times.
Proof (step 2) : repetitivity

Any finite patch is contained up to a translation in any ball of large enough radius?

- Finite patch $\mathcal{P} = [\pi(z_k), i_k] (1 \leq k \leq \ell)$
- Allow translations of $\mathcal{P}$ in $\Gamma_{sr}$.
  - $0 \leq \langle z_k, v_\beta \rangle < (1 - \varepsilon_k) \langle e_{i_k}, v_\beta \rangle$ for $1 \leq k \leq \ell$
  - Set $\varepsilon := \frac{1}{2} \min \varepsilon_k \langle e_{i_k}, v_\beta \rangle$.
  - $x \in \mathbb{H}_c + [0, \varepsilon]v_\beta \cap \mathbb{Z}^n \implies \pi(x) + \mathcal{P} \subset \Gamma_{sr}$

- Find appropriate $x$ in any large enough ball
  - Let $\gamma \in \mathbb{H}_c$.
    - $R = 2 \times \text{width(face)}$ Then $B(\gamma, R)$ contains $z \in \Gamma_{srs}$
    - Let $x_0$ such that $\eta = \langle x_0, v_\beta \rangle < \varepsilon/2$.
      Split $\mathbb{H}_c + [0, \varepsilon]v_\beta$ into slides of height $\varepsilon/2$.
      Since $\eta < \varepsilon/2$, there exists $m_k \in \mathbb{Z}$ such that
      $m_k x_0 + (\Gamma_{sr} \cap \mathbb{H}_c + [k \varepsilon/2, (k + 1)\varepsilon/2]v_\beta) \subset \mathbb{H}_c + [0, \varepsilon]v_\beta$
    - Conclusion: One of the points $z + m_k x_0$ lies in $\mathbb{H}_c + [0, \varepsilon]v_\beta$ and $\pi(z + m_k x_0) + \mathcal{P}$ appears in $\Gamma_{srs}$.
      $\mathcal{P}$ appears in any ball with radius $R_1 = R + \max \|m_k z_k\|$
Proof (intermediate lemma): effect of inflation on intersections

Tiles with centers in $\tilde{E}^N_1[\eta, j^*]$ are measurably disjoint

Proposition

If $[\gamma_1, i_1^*], [\gamma_2, i_2^*] \in \tilde{E}^N_1[\eta, j^*]$ then

$$\mu((T(i_1) + \gamma_1) \cap (T(i_2) + \gamma_2) = 0.$$

- Revisit GIFS equation by using the GIFS map.
  $$T(j) + \eta = \bigcup_{[\gamma,i^*] \in \tilde{E}^N_1[\eta,j^*]} h(T(i) + \gamma).$$

- Iterate
  $$T(j) + \eta = \bigcup_{[\gamma,i^*] \in \tilde{E}^N_1[\eta,j^*]} h^N(T(i) + \gamma).$$

- GIFS equation: intersections have zero measure
  $$[\gamma_1, i_1^*], [\gamma_2, i_2^*] \in \tilde{E}^N_1[\eta, j^*] \implies \mu(h^N(T(i_1) + \gamma_1) \cap h^N(T(i_2) + \gamma_2) = 0$$
  Hence $\mu(T(i_1) + \gamma_1) \cap (T(i_2) + \gamma_2) = 0.$
Proof (step 3) : multiple tiling

Assume that the collection is not a multiple tiling.

• Consider two balls with different covering degrees :
  • \(\gamma_1, \gamma_2 \in \mathbb{H}_c\),
  • \(P_1, P_2\) patches with different cardinal \(\ell_1 < \ell_2\)
    \(B_\varepsilon(\gamma_j) \subset \bigcap_{[\gamma, i] \in P_j} (\mathcal{I}(i) + \gamma)\).

• Expand the tiling.
  • \(h^{-m}B_\varepsilon(\gamma_1)\) is covered by \(\ell_1\) tiles in \(h^{-m}\mathcal{I}\).
  • From intermediate step, each tile in \(h^{-m}\mathcal{I}\) can be decomposed into disjoint tiles of \(\mathcal{I}\).
    \(h^{-m}B_\varepsilon(\gamma_1)\) is covered by exactly \(\ell_1\) tiles in \(\mathcal{I}\).

• Use repetitivity: For large enough \(m\), \(h^{-m}B_\varepsilon(\gamma_1)\) contains a translated copy of the patch \(P_2\).
  A ball in \(h^{-m}B_\varepsilon(\gamma_1)\) should be covered by \(\ell_2\) tiles. Contradiction.
Section 4: other tilings and Pisot conjecture
Another tiling: lattice multiple tiling

New discrete surface: a regular one!

Lattice translation set

\[ \Gamma_{lat} = \left\{ [\gamma, i^*] \in \pi(\mathbb{Z}^n) \times A \mid \gamma \in \sum_{k=2}^{n} \mathbb{Z}(\pi(e_k) - \pi(e_1)) \right\}. \]

Proposition (Canterini&Siegel’01)

Let \( \sigma \) be a primitive unit Pisot substitution that satisfies the strong coincidence condition. The lattice translation set \( \Gamma_{lat} \) is a Delaunay set that provides a multiple tiling for the subtiles

\[ \mathbb{H}_c = \bigcup_{[\gamma, i^*] \in \Gamma_{lat}} (\mathcal{T}(i) + \gamma). \]
Relations and Pisot conjecture

Theorem (Rao&Ito’06, Barge&Kwapisz’06)

Let \( \sigma \) be a unit substitution of Pisot type. The self-replicating multiple tiling is a tiling if and only if the lattice multiple tiling is also a tiling.

(No more true when there are other expanding eigenvalues that are not Galois conjugates of \( \beta \))

- Pisot conjecture: The tiling property is always true!
  No proof. Only conditions for tilings related to several graphs.
- Lattice tiling \( \rightarrow \) spectral study of substitutions.
  Self-replicating substitution tiling \( \rightarrow \) approximation of irrational planes.

In the next lectures we will focus on SRS tiling...